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Moufang quadrangles: A unifying algebraic structure, and some results on exceptional quadrangles

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Structures are the weapons of the mathematician.

– Bourbaki –

Preface

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Introduction

Moufang polygons are certain geometrical objects which play a very important role in the theory of Tits-buildings. In order to understand this connection, we will start by giving a brief introduction to this theory. There are several good books on this subject; see for example Brown's book [9], Ronan's book [39] or Tits's original lecture notes [44].

Let I be a set. A *Coxeter diagram* over I is a symmetric matrix $M = (m_{ij})_{i,j \in I}$ such that $m_{ij} \in \{2, 3, \dots\} \cup \{\infty\}$ for all $i \neq j \in I$ and such that $m_{ii} = 1$ for all $i \in I$. A Coxeter diagram can also be represented as an edge-labeled graph with vertex set I and with edge set consisting of all unordered pairs $\{i, j\}$ such that $m_{ij} \geq 3$, together with the label m_{ij}. (The label $m_{ij} = 3$ is usually suppressed, and the label $m_{ij} = 4$ is often represented by a double edge connecting i and j.) A Coxeter diagram is called *irreducible* if its corresponding graph is connected.

Let M be a Coxeter diagram over I. A *Coxeter system* of *type* M is a pair (W, S), where W is a group and $S = \{s_i \mid i \in I\}$ is a set of generators of W such that

$$W = \langle s_i \mid (s_i s_j)^{m_{ij}} = 1 \text{ for all } i, j \in I \rangle$$

is a presentation for W; W is then called a *Coxeter group*. Given a subset $J \subseteq I$, then we put $S_J := \{s_j \mid j \in J\}$ and $W_J := \langle S_J \rangle$. If (W, S) is a Coxeter system, then we have a natural *length* function from W into the set of natural numbers, which assigns to each element of W the length of a shortest representation as a product of elements of S; the length of $w \in W$ will be denoted by $\ell(w)$. We define the *distance* between two elements $x, y \in W$ as $\operatorname{dist}(x, y) := \ell(x^{-1}y)$. Two elements x and y of W are called *adjacent* if $\operatorname{dist}(x, y) = 1$, that is, if $x^{-1}y \in S$.

For a given Coxeter diagram M, there exists up to isomorphism only one Coxeter system (W, S) of type M. The diagram is called *spherical*[1] if the corresponding Coxeter group W is finite. In that case, there exists a

[1] The spherical Coxeter diagrams have been classified by H.S.M. Coxeter in [12]; they always belong to the following famous list: A_n, $B_n = C_n$, D_n, E_6, E_7, E_8, F_4, $G_2^{(m)}$, H_3, H_4.

unique longest element $r \in W$ with respect to the length function ℓ; this element r is always an involution.

Let (W, S) be a Coxeter system of type M. A *root* of (W, S) is a subset α of the form
$$\{w \in W \mid \mathrm{dist}(w, x) < \mathrm{dist}(w, y)\}$$
for some ordered pair (x, y) of adjacent elements of W.

Let I be a set, let M be a Coxeter matrix over I and let (W, S) be a Coxeter system of type M. A *building* of *type* M is a pair $\mathcal{B} = (\mathcal{C}, \delta)$ where $\mathcal{C}$ is a set and where $\delta : \mathcal{C} \times \mathcal{C} \to W$ is a *distance function* satisfying the following axioms where $x, y \in \mathcal{C}$ and $w = \delta(x, y)$:

B$_1$. $w = 1$ if and only if $x = y$;

B$_2$. If $z \in \mathcal{C}$ is such that $\delta(y, z) = s \in S$, then $\delta(x, z) = w$ or ws, and if we have moreover that $\ell(ws) = \ell(w) + 1$, then $\delta(x, z) = ws$;

B$_3$. If $s \in S$, there exists $z \in \mathcal{C}$ such that $\delta(y, z) = s$ and $\delta(x, z) = ws$.

Given a building $\mathcal{B} = (\mathcal{C}, \delta)$, then the elements of $\mathcal{C}$ are called *chambers*. We call the group W the *Weyl group*, the pair (W, S) the *Weyl system* and the map δ the W-*distance function* of the building $\mathcal{B}$. The cardinality of I is called the *rank* of the building $\mathcal{B}$. A building $\mathcal{B}$ of type M is called *irreducible* if the Coxeter diagram M is irreducible. Given a set $J \subseteq I$ and $x \in \mathcal{C}$, the *J-residue* of x is the set
$$R_J(x) := \{y \in \mathcal{C} \mid \delta(x, y) \in W_J\} .$$
Each J-residue is a building of type M_J with the distance function induced by δ. For every $i \in I$, we define an *i-panel* as a set of the form
$$\{x\} \cup \{y \in \mathcal{C} \mid \delta(x, y) = s_i\}$$
for some chamber $x \in \mathcal{C}$. A building $\mathcal{B}$ is called *thick* if each panel contains at least three chambers.

A building $\mathcal{B} = (\mathcal{C}, \delta)$ is called *spherical* if the corresponding Coxeter diagram M is spherical. Every residue of a spherical building is again a spherical building.

A map π from a subset X of W to $\mathcal{C}$ will be called an *isometry* if and only if $\delta(x^\pi, y^\pi) = x^{-1}y$ for all $x, y \in X$. An *apartment* of $\mathcal{B}$ is the image of an isometry from W to $\mathcal{C}$. A *root* of $\mathcal{B}$ is the image of an isometry from a root of (W, S) to $\mathcal{C}$. Let α be a root of $\mathcal{B}$. The *boundary* of α, denoted by $\partial\alpha$, is the set of all panels of $\mathcal{B}$ which contain exactly one chamber in α. The *interior* of α, denoted by α°, is the set of all panels of $\mathcal{B}$ which contain two chambers in α. Now let $\mathcal{B}$ be a spherical building. The *root group* of $\mathcal{B}$ corresponding to α, denoted by U_α, is defined as the group

$$U_\alpha := \{g \in \mathrm{Aut}(\mathcal{B}) \mid g \text{ acts trivially on each panel of } \alpha^\circ\} \ .$$

We now come to the important notion of a *Moufang spherical building*. A spherical building $\mathcal{B}$ is called *Moufang* if it is thick, irreducible, of rank at least two and if for each root α of $\mathcal{B}$, the root group U_α acts transitively on the set of apartments containing α. We then also say that $\mathcal{B}$ satisfies the Moufang condition.

We can now define a *generalized polygon* as a spherical building $\mathcal{B}$ of rank 2. Since $\mathcal{B}$ is spherical, the number $n := m_{12}$ is finite, and the Weyl group W of the building $\mathcal{B}$ is the dihedral group D_{2n}. The building $\mathcal{B}$ is then called a *generalized n-gon*. A *Moufang polygon* is a generalized polygon which satisfies the Moufang condition.

The following astonishing result, first proved by J. Tits in [47], gave rise to the study of Moufang polygons.

> *Every thick irreducible spherical building of rank at least three is Moufang. Moreover, every irreducible residue of rank at least two of a Moufang spherical building is also a Moufang spherical building.*

The notion of a *generalized polygon* is actually older than the one of a building. The terminology appeared for the first time in the appendix of a long and difficult paper of 1959 "*Sur la trialité et certains groupes qui s'en déduisent,*" by J. Tits, in which he discovered the simple groups of type ${}^3D_4$ by classifying the trialities with at least one absolute point of a D_4-geometry. The methods he used were of a very geometric nature, and it should not be surprising that the corresponding geometries came into play – this was the official birth of the generalized hexagons. It should be mentioned that generalized quadrangles were already studied *avant la lettre*, for example as the point-line-geometries arising from a non-singular quadric of Witt index 2, or arising from a symplectic polarity in $\mathsf{PG}(3, K)$. Also generalized triangles, which are the projective planes, had already extensively been studied before; see, for example, [38] and [22].

There is no hope that generalized polygons can be classified without requiring some additional assumptions, since there exist free constructions of generalized n-gons for all n [47]. But the fact that *Moufang* polygons could be classified was already conjectured by J. Tits in his 1974 Lecture Notes[2],

[2]In his addenda on Moufang polygons, J. Tits writes "The progress recently made on that conjecture gives reasonable hope that it might be established soon". It's a matter of taste whether you consider 28 years later to be "soon"...

and became more apparent in [45]. The fact that Moufang n-gons exist for $n \in \{3,4,6,8\}$ only was already proved shortly after this conjecture [46, 48, 54]. As Tits already observed in the very beginning, the classification of *finite* Moufang polygons easily follows from the papers by P. Fong and G. Seitz on the classification of finite BN-pairs of rank 2 [18, 19].

In 1996, Jacques Tits and Richard Weiss decided to write down a proof of the classification of Moufang polygons, in a way which is completely elementary, and entirely independent of the work by Fong and Seitz. This classification is now completed, and has recently appeared in the form of a book "Moufang Polygons" [52]. Surprisingly, a new class of Moufang quadrangles was found by R. Weiss during this classification process; these quadrangles were then recognized by B. Mühlherr and H. Van Maldeghem in the buildings of type F_4 [34].

For $n \in \{3,6,8\}$, the proof of the classification theorem consists of two nicely separated parts: it is first shown that every Moufang n-gon can be parametrized by a certain algebraic structure, and then these algebraic structures are classified. In the case $n = 3$, these algebraic structures are the alternative division rings; in the case $n = 6$, they are the anisotropic cubic norm structures (a subclass of the Jordan division algebras); and in the case $n = 8$, they are the so-called octagonal systems.

For $n = 4$, no such a uniform algebraic structure was known, and the proof of the classification of Moufang quadrangles does not consist of the division into these two parts. Instead, there are six different classes of Moufang quadrangles, and even then, this distinction is not apparent in the two cases which lead to the exceptional quadrangles, that is, those of type E_k with $k \in \{6,7,8\}$ and those of type F_4.

The second chapter of this thesis aims to provide such a uniform algebraic structure for Moufang quadrangles. We introduce this structure as a pair of two groups which act on each other, and which satisfy a certain list of axioms, and we show that every Moufang quadrangle can be parametrized by such a structure which we therefore call a *quadrangular system*. We then classify these structures without going back to the Moufang quadrangles from which they arise, thereby also providing a new proof for the classification of Moufang quadrangles. Apart from giving a new proof of the classification, these quadrangular systems ought to give more insight into the structure of the Moufang quadrangles.

In chapter 3, we give an application of these quadrangular systems. It is needless to say that the study of the automorphism group G of a geometry is an interesting problem. On the other hand, for a building, the root groups play a very important role, and the group $G^\dagger$ generated by all the root groups is an important normal subgroup of the automorphism group. In particular, it turns out to be useful to know the structure of the quotient $G/G^\dagger$. In [52], this problem is solved for all Moufang triangles and Moufang octagons, and some subclasses of Moufang quadrangles and Moufang hexagons. The cases which are left open are the two cases of the exceptional Moufang quadrangles (those of type E_k and those of type F_4), and the two cases of the exceptional Moufang hexagons, that is, those which are parametrized by an exceptional Jordan division algebra (which is of dimension 27 over the base field). We will solve the problem for the case of the exceptional Moufang quadrangles of type F_4; we will make very extensive use of the structure of the quadrangular systems of type F_4.

In the last chapter, we take a closer look at the exceptional Moufang quadrangles of type E_6, E_7 and E_8. It is already clear from the construction that the even Clifford algebra of the quadratic form which determines such a Moufang quadrangle plays a prominent role in the understanding of the structure of these exceptional quadrangles. We go one step further, by showing that those quadratic forms are *completely* characterized by the structure of their even Clifford algebra. The results of chapter 4 are used in the appendix of [52] to show that the algebraic group-theoretical condition for the existence of a Moufang quadrangle of type E_k corresponds to the elementary existence condition in terms of norm splittings.

Contents

1 Preliminaries

In this very first chapter, we will recall a whole bunch of definitions and facts which are more or less “standard”. In the first four sections, we give the necessary geometric background; in the remaining five sections, we introduce some important algebraic notions.

1.1 Graphs

A *graph* Γ is a pair $(V(\Gamma), E(\Gamma))$, where $V(\Gamma)$ is a set and where $E(\Gamma)$ is a subset of the set of all unordered pairs of $V(\Gamma)$. The elements of $V(\Gamma)$ are called *vertices*, the elements of $E(\Gamma)$ are called *edges*.

Two vertices $x, y \in V(\Gamma)$ are called *adjacent* if and only if $\{x, y\} \in E(\Gamma)$. The set of all vertices which are adjacent to some fixed vertex x, together with the element x itself, is called the *neighborhood of x* and will be denoted by Γ_x. A graph Γ is called *thick* if and only if $|\Gamma_x| \geq 3$ for all $x \in V(\Gamma)$.

A *k-path* or a *path of length k* ($k \in \mathbb{N}$) is a sequence of $k+1$ vertices $(x_0, x_1, \ldots, x_k)$ such that x_i is adjacent to x_{i-1} for all $i \in \{1, \ldots, k\}$ and such that $x_{i+1} \neq x_{i-1}$ for all $i \in \{1, \ldots, k-1\}$. A *circuit of length k* is a k-path $(x_0, x_1, \ldots, x_k)$ with $k > 2$ such that $x_0 = x_k$.

The *distance* between two vertices x and y, denoted by $\mathrm{dist}(x, y)$, is the minimal length of a path joining x and y; the distance is defined to be ∞ if there is no such path. The *diameter* of a graph Γ, denoted by $\mathrm{diam}(\Gamma)$, is defined as the maximal distance between two vertices of Γ. A graph Γ is called *connected* if and only if $\mathrm{diam}(\Gamma)$ is finite. “Having finite distance” is an equivalence relation on the vertices; its equivalence classes are called the *connected components* of the graph Γ. The *girth* of a graph Γ is the length of a shortest circuit of Γ; it is defined to be ∞ if Γ has no circuits. A *tree* is a connected graph with girth ∞, and every graph with girth ∞ is the disjoint union of trees (that is, each of its connected components is a tree).

A graph Γ is called *bipartite* if it does not have circuits of odd length. Equivalently, Γ is bipartite if $V(\Gamma)$ can be partitioned into two disjoint sets such that every edge has one vertex in each of the partitions. We also say that two vertices are of the same *type* with respect to such a partitioning if and only if they belong to the same partition. If Γ is connected, then this partitioning is unique.

1.2 Geometries

A *geometry (of rank* 2*)* is a triple $\Gamma = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ where $\mathcal{P}$ and $\mathcal{L}$ are two disjoint sets, called *points* and *lines*, and where $\mathbf{I} \subseteq \mathcal{P} \times \mathcal{L}$, called the *incidence relation*. The *dual geometry* of Γ, denoted by Γ^D, is the geometry $(\mathcal{L}, \mathcal{P}, \mathbf{I}')$, where $L\mathbf{I}'p$ if and only if $p\mathbf{I}L$. We usually consider $\mathbf{I}$ as a symmetric relation, and we will not distinguish between $\mathbf{I}$ and $\mathbf{I}'$.

A *subgeometry* of a geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ is a geometry $(\mathcal{P}', \mathcal{L}', \mathbf{I}')$ where $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{L}' \subseteq \mathcal{L}$, and $\mathbf{I}' = \mathbf{I} \cap (\mathcal{P}' \times \mathcal{L}')$. If we consider geometries satisfying certain axioms, then we are usually interested in the subgeometries which satisfy the same axioms.

A point or a line is called *thick* if it is incident with at least 3 elements. If all points and lines of a geometry Γ are thick, then we say that Γ is thick. A geometry Γ is called *finite* if $\mathcal{P}$ and $\mathcal{L}$ are finite.

We can associate in a very natural way a graph to each geometry Γ, called the *incidence graph* of Γ, which we define as follows. Let $V(\Gamma) := \mathcal{P} \cup \mathcal{L}$, and let $E(\Gamma)$ denote the set of incident point-line pairs of Γ. Then the graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$ is the incidence graph of Γ; it is always bipartite. Moreover, every bipartite graph is the incidence graph of some geometry Γ.

The *distance* between two elements of a geometry Γ is defined as the distance between the corresponding vertices in the incidence graph. In particular, the distance between two elements of the same type (i.e., two points or two lines), if finite, will always be even. A geometry Γ will be called *connected* if and only if its incidence graph is connected.

Now let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ and $\Gamma' = (\mathcal{P}', \mathcal{L}', \mathbf{I}')$ be two geometries. An *isomorphism* or a *collineation* from Γ to Γ' is a pair (α, β), where α is a bijection from $\mathcal{P}$ to $\mathcal{P}'$ and β is a bijection from $\mathcal{L}$ to $\mathcal{L}'$, preserving incidence and non-incidence. The geometries Γ and Γ' are called *isomorphic* if and only if there exists an isomorphism from Γ to Γ'.

An *automorphism* of a geometry Γ is an isomorphism from Γ to itself. An automorphism of order 2 is called an *involution*. An *anti-isomorphism* or *duality* from Γ to Γ' is a collineation from Γ to the dual Γ'^D of Γ'. An anti-isomorphism from Γ to itself is called a *correlation* or an *anti-automorphism* of Γ. A correlation of order 2 is called a *polarity*.

All automorphisms of a geometry Γ form a group, called the *automorphism group* or *collineation group* of Γ, which is denoted by $\mathrm{Aut}(\Gamma)$. All automorphisms and anti-automorphisms of Γ also form a group, called the *correlation group* of Γ, which contains $\mathrm{Aut}(\Gamma)$ as a subgroup of index 1 or 2. We will denote the correlation group of Γ by $\mathrm{Cor}(\Gamma)$.

We now introduce a useful notation. Let Γ be a geometry, and let G be an arbitrary subgroup of $\mathrm{Aut}(\Gamma)$. Then we will denote the pointwise stabilizer of the set

$$\{v \in V(\Gamma) \mid \mathrm{dist}(x, v) \leq i\}$$

by $G_x^{[i]}$. Moreover, we define

$$G^{[i]}_{x_1,x_2,\ldots,x_k} := G^{[i]}_{x_1} \cap G^{[i]}_{x_2} \cap \cdots \cap G^{[i]}_{x_k} .$$

If $i = 0$, then we will simply write $G_{x_1,x_2,\ldots,x_k}$. Note that $G_x^{[1]}$ is the kernel of the action of G_x on Γ_x.

Throughout this thesis, we will use the following convention. If S is a group, then we define $S^* := S \setminus \{\text{neutral element}\}$. If S is a set which contains an element called "0", then we define $S^* := S \setminus \{0\}$. It will always be clear from the context which definition we mean.

1.3 Generalized Polygons

A *generalized n-gon* is a connected bipartite graph with diameter n and girth $2n$, where $n \geq 2$. If we do not want to specify the value of n, then we call this a *generalized polygon*. We will also use the terminology *generalized triangle*, *generalized quadrangle*, *generalized hexagon*, and so on, instead of generalized 3- , 4- , 6-gon, respectively. This definition has been introduced in 1959 by Jacques Tits in the appendix of [42]. One of the main recent references on generalized polygons is [53].

As explained in section 1.2, every bipartite graph can be considered as a geometry. From the geometric point of view, a generalized polygon is a geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ satisfying the following two axioms. See, for example, [53, 1.3.1, 1.3.5 and 1.3.6] for the equivalence of these definitions.

GP1. If $x, y \in \mathcal{P} \cup \mathcal{L}$ and $\mathrm{dist}(x, y) = k < n$, then there exists a unique k-path from x to y.

GP2. For every $x \in \mathcal{P} \cup \mathcal{L}$, we have that $\sup\{\mathrm{dist}(x, y) \mid y \in \mathcal{P} \cup \mathcal{L}\} = n$.

Here is another equivalent definition, which explains the terminology.

GP1′. Γ does not contain ordinary k-gons (as a subgeometry), for every $k \in \{2, \dots, n-1\}$.

GP2′. Every two elements $x, y \in \mathcal{P} \cup \mathcal{L}$ are contained in an ordinary n-gon of Γ.

Every ordinary n-gon in a generalized n-gon Γ is called an *apartment* of Γ. The subgraph spanned by the vertices of an n-path in Γ is called a *half-apartment* or a *root* of Γ.

We will now briefly explain the geometric structure of a generalized n-gon for the smallest values of n.

$n = 2$. A generalized 2-gon is a geometry in which every point is incident with every line, that is, $\mathbf{I} = \mathcal{P} \times \mathcal{L}$.

$n = 3$. A generalized triangle is exactly the same thing as a (possibly degenerate) projective plane. Every two points are incident with exactly one line, and every two lines are incident with exactly one point. Projective planes have been extensively studied in [38] and [22].

$n = 4$. A generalized quadrangle is a geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ satisfying the following two axioms.

GQ1. For every non-incident point-line pair (p, L), there is a unique point q and a unique line M such that $p\mathbf{I}M\mathbf{I}q\mathbf{I}L$.

GQ2. Every point is incident with at least 2, but not with all, lines; every line is incident with at least 2, but not with all, points.

One of the most important contributions to the theory of *finite* generalized quadrangles is [35].

It is possible to give similar descriptions for other values of n as well, but we will omit this.

Note that generalized n-gons do exist for all $n \geq 2$; a free construction starting from a so-called partial n-gon has been obtained by J. Tits [47]. However, we have the following famous theorem of Feit and Higman.

Theorem 1.3.1. *Finite thick generalized n-gons exist for $n \in \{2, 3, 4, 6, 8\}$ only.*

Proof. See [17]. □

We will now mention some basic properties about generalized n-gons.

Theorem 1.3.2. *Let Γ be a thick generalized n-gon and let $G \leq \mathrm{Aut}(\Gamma)$. Then*

(i) *Every* $(n+1)$*-path is contained in a unique apartment ;*

(ii) $G^{[1]}_{x_0} \cap G_{x_0,\dots,x_n} = G_{x_0,\dots,x_n} \cap G^{[1]}_{x_n}$ *for every* n*-path* $(x_0, \dots, x_n)$ *;*

(iii) $G^{[1]}_{x_0,x_1} \cap G_{x_0,\dots,x_n} = 1$ *for every* n*-path* $(x_0, \dots, x_n)$ *;*

(iv) $G^{[1]}_{x_0,\dots,x_k} = 1$ *for every* k*-path* $(x_0, \dots, x_k)$ *with* $k \geq n-1$ *.*

Proof. See [52, (3.2), (3.5), (3.7) and (3.8)]. □

1.4 Moufang Polygons

Let Γ be a thick generalized n-gon with $n \geq 3$, and let γ be an $(n-2)$-path of Γ. An automorphism g of Γ is called a *root elation*, a γ*-elation* or simply an *elation* if and only if g fixes all elements of Γ which are incident with at least one element of γ.

Now consider a root $\alpha = (x_0, x_1, \dots, x_{n-1}, x_n)$, and let γ denote the sub-$(n-2)$-path $(x_1, \dots, x_{n-1})$. Then the group U_α of all γ-elations (called a *root group*) acts semi-regularly on the set of vertices incident with x_0 but different from x_1. If U_α acts transitively on this set (and hence regularly), then we say that α is a *Moufang root*. It turns out that this definition is independent of the choice of x_0 and x_n, and independent of the choice of the direction of the n-path α. Moreover, it turns out that α is a Moufang root if and only if U_α acts regularly on the set of apartments through α. A *Moufang* n*-gon* is a generalized n-gon for which every root is Moufang. We then also say that Γ satisfies the *Moufang condition*. The group generated by all the root groups is sometimes called the *little projective group* of Γ.

Let us assume from now on that Γ is a thick Moufang n-gon for some $n \geq 3$, and let us fix an apartment Σ which we will label by the integers modulo $2n$ in a natural way, that is, such that $i+1 \in \Gamma_i$ and $i+2 \neq i$ for all integers i.

For every root $\alpha_i := (i, i+1, \dots, i+n)$ in Σ, we define $U_i := U_{\alpha_i}$. Note that all root groups of Γ are non-trivial since Γ is thick and satisfies the Moufang condition. Furthermore, we define

$$U_{[i,j]} := \begin{cases} \langle U_i, U_{i+1}, \dots, U_j \rangle & \text{if } i \leq j < i+n \text{ ;} \\ 1 & \text{otherwise .} \end{cases}$$

Theorem 1.4.1. *The groups* U_i *satisfy the following properties.*

(i) $[U_i, U_j] \leq U_{[i+1,j-1]}$ *for all* $j \in \{i+1, \dots, i+n-1\}$ *;*

(ii) *For every integer* i*, the product map from* $U_i \times U_{i+1} \times \cdots \times U_{i+n-1}$ *to* $U_{[i,i+n-1]}$ *is bijective .*

Proof. See [52, (5.5) and (5.6)]. □

Thanks to this theorem, we can use the following notation. Let $a_i \in U_i$ and $a_j \in U_j$, with $j \in \{i+2, \dots, i+n-1\}$. For each k such that $i < k < j$, we set

$$[a_i, a_j]_k = a_k \ ,$$

where a_k is the unique element of U_k appearing in the factorization of $[a_i, a_j] \in U_{[i+1, j-1]}$.

The following property will allow us to identify root elations with automorphisms of certain subgroups of $\mathrm{Aut}(\Gamma)$.

Lemma 1.4.2. *U_i acts faithfully on $U_{[i+1,i+n-1]}$ and on $U_{[i-n+1,i-1]}$ for all i.*

Proof. See [52, (6.5)]. □

We will now concentrate on the groups $U_1, \dots, U_n$. Let $U_+ := U_{[1,n]} = \langle U_1, \dots, U_n \rangle$. Let ϕ denote the map from $V(\Sigma) = \{1, \dots, 2n\}$ to the set of subgroups of U_+ given by

$$\phi(i) := \begin{cases} U_{[1,i]} & \text{if } 1 \le i \le n \ ; \\ U_{[i-n,n]} & \text{if } n+1 \le i \le 2n \ . \end{cases}$$

We can now define a graph Ξ as follows. Let

$$V(\Xi) := \{(i, \phi(i)g) \mid i \in V(\Sigma), g \in U_+\} \ ,$$

where $\phi(i)g$ is the right coset of the subgroup $\phi(i)$ containing g. Let

$$E(\Xi) := \{\{(i,R),(j,T)\} \mid |i-j| = 1, R \cap T \neq \emptyset\} \ ,$$

where the expression $|i-j| = 1$ is to be evaluated modulo $2n$. Then $\Xi := (V(\Xi), E(\Xi))$ is a graph which is completely determined by the $(n+1)$-tuple

$$(U_+, U_1, U_2, \dots, U_n) \ .$$

Observe that there is a natural action of U_+ on Ξ, given by $(i,R)^g = (i, Rg)$ for all $(i,R) \in V(\Xi)$ and all $g \in U_+$.

The following theorem is fundamental for the classification of the Moufang polygons.

Theorem 1.4.3. $\Xi \cong \Gamma$. *In particular, the Moufang n-gon Γ is completely determined by the $(n+1)$-tuple*

$$(U_+, U_1, U_2, \dots, U_n) \ .$$

Proof. See [52, Chapter 7]. □

It is clear that not every $(n+1)$-tuple $(U_+, U_1, U_2, \dots, U_n)$ will give rise to a Moufang n-gon. In particular, such an $(n+1)$-tuple will have to satisfy the statements of Theorem 1.4.1:

$\mathcal{M}_1$. $[U_i, U_j] \le U_{[i+1,j-1]}$ for $1 \le i < j \le n$.

$\mathcal{M}_2$. The product map from $U_1 \times \cdots \times U_n$ to U_+ is bijective.

By Theorem 1.4.3, the graph Ξ above only depends on the $(n+1)$-tuple $(U_+, U_1, U_2, \dots, U_n)$, and not on the full automorphism group $\mathrm{Aut}(\Gamma)$ in which U_+ is contained. So let us now assume that we start with a certain group U_+ which is generated by certain non-trivial subgroups $U_1, \dots, U_n$, such that the conditions $(\mathcal{M}_1)$ and $(\mathcal{M}_2)$ hold; but we do not assume that U_+ is contained in a specific larger group. Furthermore, let us assume that Σ is a circuit of length $2n$ labeled by the integers modulo $2n$, but we do not assume that Σ is a subgraph of some specific larger graph. Then we can still construct a graph Ξ as above. We would like to know under which conditions this graph Ξ is a Moufang n-gon.

We first introduce another notation. It follows from $(\mathcal{M}_1)$ that the group U_n normalizes the group $U_{[1,n-1]}$. Let $\tilde{U}_n$ denote the subgroup of $\mathrm{Aut}(U_{[1,n-1]})$ induced by U_n. By Lemma 1.4.2, $\tilde{U}_n \cong U_n$. We will denote the unique element of $\tilde{U}_n$ corresponding to an element $a_n \in U_n$ by $\tilde{a}_n$.

Similarly, U_1 normalizes $U_{[2,n]}$, and we let $\tilde{U}_1$ denote the subgroup of $\mathrm{Aut}(U_{[2,n]})$ induced by U_1. Again by Lemma 1.4.2, $\tilde{U}_1 \cong U_1$, and we will denote the unique element of $\tilde{U}_1$ corresponding to an element $a_1 \in U_1$ by $\tilde{a}_1$.

Theorem 1.4.4. *Suppose that U_+ is a group generated by non-trivial subgroups $U_1, \dots, U_n$, such that the following axioms hold.*

$\mathcal{M}_1$. $[U_i, U_j] \le U_{[i+1,j-1]}$ *for* $1 \le i < j \le n$.

$\mathcal{M}_2$. *The product map from $U_1 \times \cdots \times U_n$ to U_+ is bijective.*

$\mathcal{M}_3$. *There exists a subgroup $\tilde{U}_0$ of $\mathrm{Aut}(U_{[1,n-1]})$ such that for each $a_n \in U_n^*$ there exists an element $\mu(a_n) \in \tilde{U}_0^* \tilde{a}_n \tilde{U}_0^*$ such that $U_j^{\mu(a_n)} = U_{n-j}$ for $1 \le j \le n-1$ and, for some $e_n \in U_n^*$, $\tilde{U}_j^{\mu(e_n)} = \tilde{U}_{n-j}$ for $j = 0$ and $j = n$.*

$\mathcal{M}_4$. *There exists a subgroup $\tilde{U}_{n+1}$ of $\mathrm{Aut}(U_{[2,n]})$ such that for each $a_1 \in U_1^*$ there exists an element $\mu(a_1) \in \tilde{U}_{n+1}^* \tilde{a}_1 \tilde{U}_{n+1}^*$ such that $U_j^{\mu(a_1)} = U_{n+2-j}$ for $2 \le j \le n$ and, for some $e_1 \in U_1^*$, $\tilde{U}_j^{\mu(e_1)} = \tilde{U}_{n+2-j}$ for $j = 1$ and $j = n+1$.*

Then the graph Ξ is a Moufang n-gon. Moreover, the automorphism groups $\tilde{U}_0$ and $\tilde{U}_{n+1}$ and the maps μ from U_n^* to $\tilde{U}_0^*\tilde{U}_n^*\tilde{U}_0^*$ and from U_1^* to $\tilde{U}_{n+1}^*\tilde{U}_1^*\tilde{U}_{n+1}^*$ are uniquely determined.

Proof. See [52, (8.11) and (8.12)]. ☐

Definition 1.4.5. Let U_+ be a group generated by non-trivial subgroups $U_1, \dots, U_n$. Then the $(n+1)$-tuple $(U_+, U_1, U_2, \dots, U_n)$ will be called a *root group sequence* if and only if $(\mathcal{M}_1)$ – $(\mathcal{M}_4)$ hold.

Remark 1.4.6. If $\Theta = (U_+, U_1, U_2, \dots, U_n)$ is a root group sequence, then $(U_+, U_n, U_{n-1}, \dots, U_1)$ is a root group sequence as well; it is called the *opposite* of Θ, and is denoted by Θ^{op}.

We finally take one step further back. Suppose that some non-trivial groups $U_1, \dots, U_n$ are given (for some $n \geq 3$), but not the larger group U_+. Let $W := U_1 \times \cdots \times U_n$. For $i, j \in \{1, \dots, n\}$, let

$$U_{[i,j]} := \{(a_1, \dots, a_n) \in W \mid a_k = 1 \ \text{ if } k < i \text{ or } k > j\} .$$

For each $i \in \{1, \dots, n\}$, we will identify U_i with the subset $U_{[i,i]}$ of W. Suppose that for each $i, j \in \{1, \dots, n\}$ we have a map ξ_{ij} from $U_i \times U_j$ to $U_{[i+1,j-1]}$. Let $\mathcal{R}$ be the set consisting of the relations

$$[a_i, a_j] = \xi_{ij}(a_i, a_j)$$

for all $i, j \in \{1, \dots, n\}$ and all $a_i \in U_i$ and $a_j \in U_j$. We would like to know under which conditions we can define a multiplication on W extending the multiplication on the individual U_i so that W becomes a group fulfilling conditions $(\mathcal{M}_1)$ and $(\mathcal{M}_2)$ in which the relations $\mathcal{R}$ hold. If such a group structure exists, then products can be calculated using only the structure of the individual U_i and the relations $\mathcal{R}$. This implies that the group structure, if it exists, is unique. To show that such a group structure exists, we try to define a group structure on $U_{[i,j]}$ for all $i, j \in \{1, \dots, n\}$ with $j - i = k$, starting with $k = 1$, and proceeding inductively. For $k = 1$, we can simply make $U_{[i,j]}$ into the direct product $U_i \times U_j$ since $U_{[i+1,j-1]}$ is trivial. We now suppose that $k \in \{2, \dots, n-1\}$ and impose the following conditions inductively:

$\mathcal{A}_k$. For all $i, j \in \{1, \dots, n\}$ with $j - i = k$ and for all $a_i, b_i \in U_i$ and $a_j \in U_j$, the equation

$$\xi_{ij}(a_i b_i, a_j) = \xi_{ij}(a_i, a_j)^{b_i} \xi_{ij}(b_i, a_j)$$

holds in the group $U_{[i,j-1]}$.

$\mathcal{B}_k$. For all $i, j \in \{1, \dots, n\}$ with $j - i = k$ and for all $a_i \in U_i$ and $a_j, b_j \in U_j$, the equation

$$\xi_{ij}(a_i, a_j b_j) = \xi_{ij}(a_i, b_j)\xi_{ij}(a_i, a_j)^{b_j}$$

holds in the group $U_{[i+1,j]}$.

$\mathcal{C}_k$. For all $i, j \in \{1, \dots, n\}$ with $j - i = k$ and for all $a_i \in U_i$, $a_j \in U_j$ and $c \in U_{[i+1,j-1]}$, the equation

$$c^{\xi_{ij}(a_i,a_j)} = c^{a_i^{-1}a_j^{-1}a_i a_j}$$

holds, where the right hand side is evaluated by using the action of U_i and U_j on $U_{[i+1,j-1]}$ obtained from the group structure on $U_{[i,j-1]}$ and $U_{[i+1,j]}$ which is known by the induction hypothesis.

Theorem 1.4.7. *Suppose that some non-trivial groups $U_1, \dots, U_n$ are given (for some $n \geq 3$), together with the relations $\mathcal{R}$ as above, and suppose that the conditions $(\mathcal{A}_k)$, $(\mathcal{B}_k)$ and $(\mathcal{C}_k)$ hold for all $k \in \{2, \dots, n-1\}$. Then there is a unique group structure on $W = U_1 \times \cdots \times U_n$ such that the relations $\mathcal{R}$ hold and such that the embeddings $U_i \hookrightarrow W$ for $i \in \{1, \dots, n\}$ are homomorphisms. This group and its subgroups $U_1, \dots, U_n$ fulfill conditions $(\mathcal{M}_1)$ and $(\mathcal{M}_2)$.*

Proof. See [52, (8.13)]. □

We end this section by translating the notion of (anti-)isomorphisms and automorphisms of Moufang polygons in terms of their root group sequences.

Theorem 1.4.8. *Let Γ and Γ' be two Moufang n-gons for some $n \geq 3$, let Σ (in Γ) and Σ' (in Γ') be apartments labeled by the integers and let U_i and U_i' for $i \in \{1, \dots, n\}$ be the root groups with respect to Σ and Σ'. Let β be the map from Σ to Σ' such that $\beta(i) = i'$ for all i. Suppose that α is an isomorphism from U_+ to U_+' mapping U_i to U_i' for all $i \in \{1, \dots, n\}$. Then β extends uniquely to an isomorphism from Γ to Γ' inducing α on U_+.*

Proof. See [52, (7.5)]. □

Using Theorem 1.4.7, we can restate this in terms of commutator relations, without mentioning the bigger groups U_+ and U_+':

Theorem 1.4.9. *Let Γ and Γ' be two Moufang n-gons for some $n \geq 3$, where Γ is defined by some groups $U_1, \dots, U_n$ and some commutator relations $\mathcal{R}$, and where Γ' is defined by some groups $U_1', \dots, U_n'$ and some commutator* *relations $\mathcal{R}'$. Then Γ and Γ' are isomorphic or anti-isomorphic if and only if there exist isomorphisms φ_i from U_i to U'_i for all $i \in \{1, \dots, n\}$ or from U_i to U'_{n+1-i} for all $i \in \{1, \dots, n\}$, such that the relations $\mathcal{R}$ are mapped onto the relations $\mathcal{R}'$.*

Now let Γ be a Moufang n-gon for some $n \geq 3$, put $G = \mathrm{Aut}(\Gamma)$, and let $G^\dagger$ be its subgroup generated by all the root groups of Γ. An important problem in the theory of Moufang polygons is to determine the structure of the quotient $G/G^\dagger$. The following theorem already shows that $G^\dagger$ must be quite large.

Theorem 1.4.10. *$G^\dagger$ acts transitively on the set of pairs (Σ, e), where Σ is an apartment of Γ and e an edge of Σ.*

Proof. See [52, (4.12)]. □

Now let $\Sigma = \{0, \dots, 2n-1\}$ be a labeled base apartment of Γ, and let H be the pointwise stabilizer of Σ. Moreover, let $H^\dagger := H \cap G^\dagger$. Then it follows from Theorem 1.4.10 that $G = HG^\dagger$, and hence

$$G/G^\dagger \cong H/H^\dagger\ . \tag{1.1}$$

The advantage of restricting to H is that we can do everything just in terms of the root group sequence now:

Theorem 1.4.11. (i) *H acts faithfully on $U_1 \times U_n$;*

(ii) *$H^\dagger = X_1 X_n$, where $X_i := \langle \mu(U_i^*)\mu(U_i^*) \rangle$ for $i \in \{1, n\}$.*

Proof. (i) See [52, (33.5)].

(ii) See [52, (33.9)].

□

1.5 Quadratic Forms

There are several very good introductory text books about quadratic forms available, for example [29] and [41].

Let K be an arbitrary commutative field (of arbitrary characteristic), and let V be an arbitrary vector space over K. A map q from V to K is called a *quadratic form* if and only if it satisfies the following two axioms.

QF$_1$. $q(tv) = t^2 q(v)$, for all $v \in V$ and all $t \in K$.

QF$_2$. The map $f : V \times V \to K$ defined by the equation

$$f(u,v) := q(u+v) - q(u) - q(v) ,$$

for all $u, v \in V$, is bilinear over K.

The map f is called the *bilinear form associated to* q. Note that f is symmetric. For every subspace V_1 of V, we define

$$V_1^{\perp} := \{v \in V \mid f(v, V_1) = 0\} .$$

Two subspaces V_1 and V_2 are called *orthogonal* if and only if $f(V_1, V_2) = 0$. The subspace $V^{\perp}$ is called the *radical of* f, and will be denoted by $\mathrm{Rad}(f)$. A quadratic form q is called *anisotropic* if $q(v) = 0$ implies $v = 0$; it is called *isotropic* otherwise; moreover, q is called *hyperbolic* if $\dim_K \ker(q) = \dim_K V/2$. The quadratic form q is called *regular* if $\mathrm{Rad}(f) = 0$; it is called *non-regular* or *singular* otherwise.

Remark 1.5.1. If W is a *regular* subspace of (V, q), then $V = W \oplus W^{\perp}$; see, for example, [41, 3.3.4].

If q is a quadratic form from a vector space V to a field K, then the triple (K, V, q) or simply the pair (V, q) if there is no confusion about the field, is called a *quadratic space*. The *dimension* of a quadratic form is the dimension of the vector space V over K on which q is defined.

If (K, V_1, q_1) and (K, V_2, q_2) are two quadratic spaces, then we can define the *orthogonal sum* of q_1 and q_2, denoted by $q_1 \perp q_2$, as the quadratic form with underlying vector space $V_1 \oplus V_2$ defined by

$$(q_1 \perp q_2)(x_1, x_2) := q_1(x_1) + q_2(x_2) ,$$

for all $x_1 \in V_1$ and $x_2 \in V_2$. In particular, we have that $\dim(q_1 \perp q_2) = \dim(q_1) + \dim(q_2)$.

Let q be an arbitrary anisotropic quadratic form from V to K, and let $c \in V^*$ be arbitrary. Then we can define the *reflection* π_c about c by the equation

$$\pi_c(v) := v - f(v,c)q(c)^{-1}c$$

for all $v \in V$. These reflections satisfy the properties $\pi_c \circ \pi_c = 1$ and $q \circ \pi_c = q$ for all $c \in V^*$.

Let q and q' be two quadratic forms from V to K. Then q' is called a *translate* of q if there exists a fixed constant $\lambda \in K^*$ such that $q'(v) = \lambda q(v)$ for all $v \in V$.

Two quadratic spaces (K_1, V_1, q_1) and (K_2, V_2, q_2) are called *isomorphic* if and only if there exists a vector space isomorphism (ψ, φ) from (K_1, V_1)

to (K_2, V_2) such that $q_2(\varphi(v)) = \psi(q_1(v))$ for all $v \in V_1$; (ψ, φ) is then called an *isomorphism* from q_1 to q_2, and we write $q_1 \cong q_2$. . If $K_1 = K_2$ and $\psi = 1$, then q_1 and q_2 are called *isometric*, and we denote this by $q_1 \simeq q_2$; the map φ is then called an *isometry* from q_1 to q_2. Two quadratic spaces (K_1, V_1, q_1) and (K_2, V_2, q_2) are called *similar* if and only if there exists a $\lambda \in K_1^*$ such that $q_2 \cong \lambda q_1$, that is, if and only if q_2 is isomorphic to a translate of q_1. If (ψ, φ) is such a vector space isomorphism from (K_1, V_1) to (K_2, V_2) for which $q_2(\varphi(v)) = \psi(\lambda q_1(v))$ for all $v \in V$, then (ψ, φ) is called a *similarity* from q_1 to q_2. If $K_1 = K_2$ and $\psi = 1$, then the map φ is then called a *similitude* from q_1 to q_2, and the constant $\lambda \in K_1^*$ is called the *multiplier* of the similitude.

If $(K_1, V_1, q_1) = (K_2, V_2, q_2) = (K, V, q)$, then we will speak about an *isometry of* q and a *similitude of* q, respectively. Not every $\lambda \in K^*$ can occur as multiplier of some similitude of q. The group of possible multipliers of similitudes of (K, V, q) is denoted by $G(K, V, q)$ or by $G(q)$ for short. An *invariant of quadratic forms* is something which is invariant under isometries of quadratic forms.

We will now define the *discriminant* $d(q)$ of a regular n-dimensional quadratic form q. First assume that $\mathrm{char}(K) \neq 2$. Let $\{e_1, \dots, e_n\}$ be an orthogonal basis of (K, V, q). (Such a basis always exists; see, for example, [41, 1.3.5].) Then we can define the *determinant of* q, denoted by $\det(q)$, as

$$\det(q) := q(e_1)q(e_2)\cdots q(e_n) ,$$

considered as an element of $K^*/(K^*)^2$. One can prove that this definition is independent of the choice of the orthogonal basis, and is hence an invariant; see, for example, [41, 1.3.17]. We can now define the *discriminant*

$$d(q) = (-1)^{n(n-1)/2} \det(q) ,$$

also considered as an element of $K^*/(K^*)^2$.

Now assume that $\mathrm{char}(K) = 2$, and let $(e_1, \dots, e_{n/2}, f_1, \dots, f_{n/2})$ be a symplectic basis of (K, V, q). (Such a basis always exists; see, for example, [41, 9.4].) Then we define the *discriminant*

$$d(q) = q(e_1)q(f_1) + \cdots + q(e_{n/2})q(f_{n/2}) ,$$

considered as an element of $K/\wp(K)$, where $\wp(K) = \{x + x^2 \mid x \in K\}$. This is also called the *Arf invariant* of q. One can prove that this definition is independent of the choice of the symplectic basis, and is hence an invariant; see, for example, [41, 9.4.2].

If $\mathrm{char}(K) \neq 2$, we will say that the discriminant is trivial if and only if $d(q) \in (K^*)^2$; if $\mathrm{char}(K) = 2$, it means that $d(q) \in \wp(K)$.

If the discriminant is non-trivial, then we can define the *discriminant extension* of q (over K). If $\mathrm{char}(K) \neq 2$, this will be the separable quadratic extension $K[X]/(X^2 - d(q))$; if $\mathrm{char}(K) = 2$, this will be the separable quadratic extension $K[X]/(X^2 + X + d(q))$.

We now introduce the notion of a *norm splitting*, which is first seen in [52, (12.9)]. First of all, observe that, if E/K is a separable quadratic extension with norm N, then N is a 2-dimensional anisotropic regular quadratic form from E (as a vector space over K) to K. (This will also allow us to speak about the *discriminant of a norm*. Note that the discriminant extension of N is exactly E/K.) We say that a $2d$-dimensional regular quadratic form $q : V \to K$ has a norm splitting, if and only if there exist constants $s_1, s_2, \dots, s_d \in K^*$ such that

$$q \simeq s_1 N \perp s_2 N \perp \cdots \perp s_d N \,.$$

The constants $s_1, s_2, \dots, s_d$ are called the *constants of the norm splitting*.

Since $d(sN) = d(N)$, for all $s \in K^*$, this is equivalent to the assumption that q has an orthogonal decomposition $q \simeq q_1 \perp q_2 \perp \cdots \perp q_d$, where each q_i is a 2-dimensional regular quadratic form with the same non-trivial discriminant. Note that a $2d$-dimensional regular quadratic form q is hyperbolic if and only if q has a decomposition $q \simeq q_1 \perp q_2 \perp \cdots \perp q_d$, where each q_i is a 2-dimensional regular quadratic form with trivial discriminant.

Remark 1.5.2. Every even dimensional regular quadratic form q has an orthogonal decomposition $q \simeq q_1 \perp q_2 \perp \cdots \perp q_d$, where each q_i is a 2-dimensional regular quadratic form. If $\mathrm{char}(K) \neq 2$, this follows from the fact that q has a diagonal form; if $\mathrm{char}(K) = 2$, this follows from the fact that q has a normal form, see, for example, [41, 9.4].

Now let (K, V, q) be an arbitrary anisotropic quadratic space with corresponding bilinear map f. An automorphism T of V is called a *norm splitting map* of q if and only if there exist constants $\alpha, \beta \in K$ with $\alpha = 0$ if $\mathrm{char}(K) \neq 2$ and $\alpha \neq 0$ if $\mathrm{char}(K) = 2$, and with $\beta \neq 0$ in all characteristics, such that

$$\begin{aligned} &q(T(v)) = \beta q(v) \,, \\ &f(v, T(v)) = \alpha q(v) \,, \\ &T(T(v)) + \alpha T(v) + \beta v = 0 \,, \end{aligned}$$

for all $v \in V$. For each norm splitting map T, we can define a corresponding norm splitting map $\overline{T}$, defined by the relation $\overline{T}(v) := \alpha v - T(v)$ for all $v \in V$. It is straightforward to check that $\overline{T}$ is a norm splitting map with the same parameters α and β as the original norm splitting map T.

Let (V, q) be an arbitrary quadratic space, and let $\epsilon \in V^*$ be an element such that $q(\epsilon) = 1$. Then (V, q, ϵ) is called a quadratic space with *base point* ϵ. For such a quadratic space, we define a map $v \mapsto \overline{v}$ by setting

$$\overline{v} := -\pi_\epsilon v = f(\epsilon, v)\epsilon - v \ ,$$

for all $v \in V$. Observe that $\overline{\overline{v}} = v$ and $q(\overline{v}) = q(v)$ for all $v \in V$.

Remark 1.5.3. Let (V, q) be an arbitrary anisotropic quadratic space, and let $\epsilon \in V^*$ be arbitrary. If we set $\lambda = q(\epsilon)^{-1}$, then $(V, \lambda q, \epsilon)$ is a quadratic space with base point ϵ.

1.6 The Brauer Group

Let K be an arbitrary commutative field. A K-vector space A is called an *algebra* over K or simply a K*-algebra* if and only if A has a multiplication which makes it into a ring with unity $1 \in A^*$, such that the map from K to A mapping every t to $t \cdot 1$ is a homomorphism from K to the center $Z(A)$ of A. We will then identify K with its image under this homomorphism. A K-algebra A is called *central* if $K = Z(A)$. An algebra A is called a *division algebra* if every non-zero element is invertible in A. A K-algebra is called *simple* if it does not contain non-trivial algebra ideals. The *degree* of a K-algebra A is the smallest integer m such that every element of A is the root of a polynomial over K of degree at most m.

For the rest of this section, we will only consider finite dimensional vector spaces and algebras; this condition will not be repeated explicitly.

Theorem 1.6.1. *Let A be an arbitrary simple K-algebra. Then $A \cong \mathrm{Mat}_n(D)$ for a suitable integer n and a suitable skew field D. Moreover, n is unique and D is unique up to isomorphism.*

Proof. This famous theorem is due to Wedderburn; see, for example, [41, 8.1.11]. □

Lemma 1.6.2. *If A and B are K-algebras, then*

$$\mathrm{Mat}_m(A) \otimes_K \mathrm{Mat}_n(B) \cong \mathrm{Mat}_{mn}(A \otimes_K B) \ .$$

Proof. This follows from [41, 8.2.3]. □

Theorem 1.6.3. *Let A and B be two central simple K-algebras. Then $A \otimes_K B$ is again a central simple K-algebra.*

Proof. See, for example, [41, 8.3.2]. □

If R is a ring, we define a ring R^{op} by $R = R^{\text{op}}$ as additive abelian groups while the product $\bullet$ in R^{op} is defined by $a \bullet b := ba$ for all $a, b \in R$. The ring R^{op} is called the *opposite* ring of R.

Theorem 1.6.4. *Let A be a central simple K-algebra. Then*

$$A \otimes_K A^{\text{op}} \cong \mathrm{Mat}_n(K) ,$$

where $n = \dim_K A$.

Proof. See, for example, [41, 8.3.4]. □

Definition 1.6.5. Let A and B be two central simple K-algebras. By Wedderburn's Theorem 1.6.1, $A \cong \mathrm{Mat}_n(D)$ and $B \cong \mathrm{Mat}_m(E)$ for some integers m and n and some skew fields D and E. Then A and B will be called *similar* if and only if $D \cong E$; we will denote this by $A \sim B$. It is obvious that similarity is an equivalence relation. The set of similarity classes will be denoted by $\mathrm{Br}(K)$. The similarity class of a central simple K-algebra A will be denoted by $[A] \in \mathrm{Br}(K)$.

It follows readily from Theorem 1.6.3 and Lemma 1.6.2 that the tensor product of central simple algebras induces a binary operator on $\mathrm{Br}(K)$. Since $A \otimes_K B \cong B \otimes_K A$ and $A \otimes_K (B \otimes_K C) \cong (A \otimes_K B) \otimes_K C$ for all central simple K-algebras A, B and C, it follows that this operator is commutative and associative. We will hence write this operator on $\mathrm{Br}(K)$ additively, that is, we will write

$$[A] + [B] := [A \otimes_K B]$$

for all central simple K-algebras A and B. It is clear that $[K]$ is a (two-sided) neutral element for this operator, hence we will write $[K] = 0$. It now follows from Theorem 1.6.4 that

$$[A] + [A^{\text{op}}] = [A \otimes_K A^{\text{op}}] = [\mathrm{Mat}_n(K)] = [K] = 0$$

for all central simple K-algebras A; hence every $[A] \in \mathrm{Br}(K)$ has an inverse $-[A] = [A^{\text{op}}] \in \mathrm{Br}(K)$.

We conclude that $\mathrm{Br}(K)$ is an abelian group, and we call it the *Brauer group*[1] of the field K.

We end this section by mentioning a related theorem which we will need in Chapter 4.

[1]In [6], Richard Brauer has shown the existence of this group whose properties give great insight into the structure of simple algebras. This group became known (to its author's embarrassment!) as the "Brauer group", and played an essential part in the proof by Brauer, Hasse and Noether of the longstanding conjecture that every rational division algebra is cyclic over its center [8].

Lemma 1.6.6. *Let E/K be a separable quadratic extension. Let D be a finite dimensional central division algebra over K.*

(i) *If E is not contained in D (i.e., if the minimal polynomial of E does not have a root in D), then $D \otimes_K E$ is a central division algebra over E;*

(ii) *If E is contained in D (i.e., if the minimal polynomial of E has a root in D), then the centralizer D_0 of E in D is a finite dimensional central division algebra over E and $D \otimes_K E \cong \mathrm{Mat}_2(D_0)$ as E-algebras.*

Proof. See [20, 11.A]. □

1.7 Quaternion Algebras

We will quickly recall the notion of a *quaternion algebra*[2]. It will be very useful to have a characteristic-free approach. Suppose that E/K is a separable quadratic extension with norm N, let σ be the non-trivial element in $\mathrm{Gal}(E/K)$, and let $\gamma \in K^*$. Following the notation in [52, 9.2], we define the quaternion algebra $Q = (E/K, \gamma)$ to be the subring of $\mathrm{Mat}_2(E)$ consisting of the matrices $\left(\begin{smallmatrix} a & \gamma b^\sigma \\ b & a^\sigma \end{smallmatrix}\right)$ for all $a, b \in E$. Let $\delta = d(N)$, then it is not very hard to see that $(E/K, \gamma) \cong \left(\frac{\delta,\gamma}{K}\right)$ if $\mathrm{char}(K) \neq 2$ and $(E/K, \gamma) \cong \left[\frac{\delta,\gamma}{K}\right)$ if $\mathrm{char}(K) = 2$; see also [28, 10.F]. Here $\left(\frac{\delta,\gamma}{K}\right)$ denotes the unique K-algebra with basis $\{1, i, j, k\}$ satisfying the relations $i^2 = \delta$, $j^2 = \gamma$, $ij = k = -ji$, and $\left[\frac{\delta,\gamma}{K}\right)$ denotes the unique K-algebra with basis $\{1, u, v, w\}$ satisfying the relations $u^2 + u = \delta$, $v^2 = \gamma$, $uv = w = vu + v$. We will denote the image of a quaternion algebra $(E/K, \gamma)$ in the Brauer group $\mathrm{Br}(K)$ by $[E/K, \gamma]$.

Theorem 1.7.1. *Let E/K be a separable quadratic extension, and let β and γ be arbitrary elements of K^*. Then, in $\mathrm{Br}(K)$,*

$$[E/K, \beta] + [E/K, \gamma] = [E/K, \beta\gamma] \ .$$

Proof. See, for example, [52, 9.5] for a characteristic-free proof of this well known fact. □

[2] The (real) quaternions $\mathbb{H}$ were discovered by William Rowan Hamilton. In 1835, at the age of 30, he had discovered how to treat complex numbers as pairs of real numbers. Fascinated by the relation between $\mathbb{C}$ and 2-dimensional geometry, he tried for many years to invent a bigger algebra that would play a similar role in 3-dimensional geometry. It took him until the 16th of October, 1843, to realize that he really needed a 4-dimensional algebra: "That is to say, I then and there felt the galvanic circuit of thought close; and the sparks which fell from it were the fundamental equations between i, j and k; exactly such as I have used them ever since." And in a famous act of mathematical vandalism, he carved these equations into the stone of the Brougham Bridge: $i^2 = j^2 = k^2 = ijk = -1$.

Lemma 1.7.2. *Let E/K be a separable quadratic extension with norm N and let $\gamma \in K^*$. Then $[E/K, \gamma] = 0$ in $\mathrm{Br}(K)$ if and only if $\gamma \in N(E)$.*

Proof. See, for example, [52, 9.4] for a characteristic-free proof of this well known fact. □

Theorem 1.7.3. *Let E/K be a separable quadratic extension, and let $\gamma \in K^*$. Then $[E/K, \gamma]$ has order at most 2 in $\mathrm{Br}(K)$.*

Proof. This follows immediately from Theorem 1.7.1 and Lemma 1.7.2, since $\gamma^2 \in N(E)$ for all $\gamma \in K^*$. □

Theorem 1.7.4. *Suppose that Q_1, Q_2 and Q_3 are three quaternion division algebras over K. If $[Q_1] + [Q_2] + [Q_3] = 0$ in $\mathrm{Br}(K)$, then Q_1, Q_2 and Q_3 have a separable quadratic subfield E in common (up to isomorphism).*

Proof. In odd characteristic, this result is essentially due to Albert [1]. In all characteristics, but without the separability condition, this has been proved almost simultaneously by Albert [3] and by Sah [40]. The full result as stated above, is due to Draxl [16]. Two new and shorter proofs have been given by Knus [27] and by Tits [50]. Very recently, an elementary proof of the separability condition has been given by T.Y. Lam [30]. □

1.8 Clifford Algebras

We will now introduce the *Clifford algebra*[3] of a quadratic form. Let (K, V, q) be a quadratic space in arbitrary characteristic, with $V \neq 0$. The *tensor algebra* of a vector space (K, V) is defined as the algebra

$$T(V) := K \oplus V \oplus (V \otimes_K V) \oplus (V \otimes_K V \otimes_K V) \oplus \cdots ,$$

and has the following universal property.

Theorem 1.8.1. *Let (K, V) be a vector space, and let A be an arbitrary associative K-algebra. Then any linear map from V to A has a unique extension to a homomorphism from $T(V)$ to A.*

Proof. See, for example, [41, 9.2]. □

[3]Introduced by William Clifford in 1876 as a generalization of Grassmann's exterior algebra; see [11].

Consider the ideal $I(V,q) := \langle u \otimes u - q(u) \cdot 1 \mid u \in V \rangle$ of $T(V)$. Then the *Clifford algebra* of q, which we denote by $C(V,q)$ or by $C(q)$ for short, is defined as the algebra

$$C(V,q) := T(V)/I(V,q) .$$

We will identify K and V with their natural image in the Clifford algebra. The multiplication of two elements $x, y \in C(q)$ will be denoted by xy in place of $x \otimes y$. With these conventions, we have that $v^2 = q(v)$ in $C(q)$, for all $v \in V$. Note that it follows from this that $uv + vu = f(u,v)$, for all $u, v \in V$. The Clifford algebra has the following universal property:

Theorem 1.8.2. *Let (K, V, q) be a quadratic space, and let A be an arbitrary associative K-algebra. Then any linear map T from V to A satisfying $T(v)^2 = q(v)$ for all $v \in V$, has a unique extension to a homomorphism from $C(V,q)$ to A.*

Proof. See, for example, [41, 9.2.2]. □

Theorem 1.8.3. *Let (K, V, q) be an n-dimensional quadratic form over K (which might or might not be regular), and let $\{e_1, \dots, e_n\}$ be a K-basis for V. Then $\dim_K C(V,q) = 2^n$, and the set*

$$\left\{ e_1^{\alpha_1} e_2^{\alpha_2} \cdots e_n^{\alpha_n} \mid \alpha_1, \dots, \alpha_n \in \{0,1\} \right\}$$

is a K-basis for $C(V,q)$.

Proof. See, for example, [41, 9.2.7] for a characteristic-free proof. □

Remark 1.8.4. In [41, 9.2.7], it is assumed that $\mathrm{char}(K) \neq 2$, but the proof of (9.2.6) and hence also of (9.2.7) holds unchanged if $\mathrm{char}(K) = 2$. In particular, the assumption that the basis is orthogonal is not needed.

The *even Clifford algebra* of q, denoted by $C_0(V,q)$ or by $C_0(q)$ for short, is defined as the subalgebra of $C(V,q)$ generated by the products uv where $u, v \in V$.

Theorem 1.8.5. *Let (K, V, q) be an n-dimensional quadratic form over K (which might or might not be regular). Then $\dim_K C_0(V,q) = 2^{n-1}$.*

Proof. This follows from Theorem 1.8.3. See, for example, [29, V.1.9]. □

Both the Clifford algebra and the even Clifford algebra play a very important role in the study of quadratic forms. Let (K, V, q) be an arbitrary *regular* quadratic space. If $\dim(q) = \dim_K V$ is even, then $C(V,q)$ is a central simple K-algebra; if $\dim(q)$ is odd, then $C_0(V,q)$ is a central simple K-algebra (see, for example, [41, 9.2.10 and 9.4.7]). In particular, there exists an invariant in $\mathrm{Br}(K)$ called the *Clifford invariant* or the *Witt invariant* which is defined as follows:

$$c(q) := \begin{cases} [C(q)] & \text{if } \dim(q) \text{ is even ;} \\ [C_0(q)] & \text{if } \dim(q) \text{ is odd .} \end{cases}$$

A technique which we will use quite often in Chapter 4 is extending the base field of a quadratic form. If we extend the scalars of (K,V,q) to some extension field E over K, then we will denote the corresponding quadratic space by $(E, V \otimes_K E, q_E)$. One can check that the functors C and C_0 behave well with respect to field extensions, that is, $C(q_E) \cong C(q) \otimes_K E$ and $C_0(q_E) \cong C_0(q) \otimes_K E$.

We finally introduce the less known concept of a *Clifford algebra with base point*, a notion which was introduced by N. Jacobson in [26]. Let (V,q,ϵ) be an arbitrary quadratic space with base point ϵ. We define an ideal $I(q,\epsilon)$ in $T(V)$ as

$$I(q,\epsilon) := \left\langle \epsilon - 1, v \otimes \overline{v} - q(v) \cdot 1 \mid v \in V \right\rangle .$$

Then the Clifford algebra with base point $C(V,q,\epsilon)$ or $C(q,\epsilon)$ for short is defined as the quotient $C(q,\epsilon) := T(V)/I(q,\epsilon)$. Again, K and V will be identified with their natural image in $C(q,\epsilon)$, and the multiplication will be denoted by juxtaposition. With these conventions, we have $\epsilon = 1$ and $v\overline{v} = q(v)$ in $C(q,\epsilon)$ for all $v \in V$; it follows from this that $u\overline{v} + v\overline{u} = f(u,v)$ in $C(q,\epsilon)$ for all $u,v \in V$.

Theorem 1.8.6. $C(q,\epsilon) \cong C_0(q)$.

Proof. See [52, (12.51)]. ☐

1.9 Three other algebraic structures

In this final section of this chapter, we will introduce three other algebraic structures which we will need later on in section 2.6 to describe some of the Moufang quadrangles.

1.9.1 Indifferent Sets

Following [52], we define an *indifferent set* as a triple (K, K_0, L_0), where K is a commutative field of characteristic 2 and K_0 and L_0 are additive subgroups of K both containing 1, such that

$$K_0^2 L_0 \subseteq L_0 \;,$$
$$L_0 K_0 \subseteq K_0 \;,$$
$$K_0 \text{ generates } K \text{ as a ring.}$$

We will just mention a few properties of indifferent sets.

Lemma 1.9.1. *Let (K, K_0, L_0) be an arbitrary indifferent set, and let L be the subring of K generated by L_0. Then*

(i) $K^2 K_0 \subseteq L K_0 \subseteq K_0$;
(ii) $K^2 L_0 \subseteq L_0$;
(iii) $L_0^2 \subseteq K_0^2 \subseteq L_0 \subseteq K_0$;
(iv) K_0^* *and* L_0^* *are closed under inverses ;*
(v) L *is a subfield of* K *;*
(vi) (L, L_0, K_0^2) *is an indifferent set, called the opposite of* (K, K_0, L_0) .

Proof. See [52, (10.2)]. □

1.9.2 Involutory Sets

As in [52], we define an *involutory set* as a triple (K, K_0, σ), where K is a field or a skew-field, σ is an involution of K, and K_0 is an additive subgroup of K containing 1 such that

$$K_\sigma \subseteq K_0 \subseteq \mathrm{Fix}_K(\sigma) \quad \text{and}$$
$$a^\sigma K_0 a \subseteq K_0 \quad \text{for all } a \in K \;,$$

where $K_\sigma := \{a + a^\sigma \mid a \in K\}$.

Note that if $\mathrm{char}(K) \neq 2$, then $K_\sigma = \mathrm{Fix}_K(\sigma)$, and hence $K_0 = K_\sigma$ as well, so the second condition is superfluous in this case. On the other hand, if $\mathrm{char}(K) = 2$, then the quotient $\mathrm{Fix}_K(\sigma)/K_\sigma$ is a right vector space over K with scalar multiplication given by

$$(x + K_\sigma) \cdot a = a^\sigma x a + K_\sigma$$

for all $x \in \mathrm{Fix}_K(\sigma)$ and all $a \in K$, so the second condition is equivalent to the assertion that K_0/K_σ is a subspace of $\mathrm{Fix}_K(\sigma)/K_\sigma$.

1.9.3 Pseudo-quadratic Forms

Let K be an arbitrary field or skew-field, let σ be an involution of K (which may be trivial), and let V_0 be a right vector space over K. A map h from $V_0 \times V_0$ to K is called a *sesquilinear form* (with respect to σ) if and only if h is additive in both variables, and $h(at, bs) = t^\sigma h(a,b)s$, for all $a, b \in V_0$ and all $t, s \in K$. A form $h : V_0 \times V_0 \to K$ is called *hermitian*, respectively *skew-hermitian*, (with respect to σ) if and only if h is sesquilinear with respect to σ and $h(a,b)^\sigma = h(b,a)$, respectively $h(a,b)^\sigma = -h(b,a)$, for all $a, b \in V_0$.

Let (K, K_0, σ) be an involutory set, let V_0 be a right vector space over K and let p be a map from V_0 to K. Then p is a *pseudo-quadratic form* on V (with respect to K_0 and σ) if there is a form h on V_0 which is skew-hermitian with respect to σ such that

$$\begin{aligned} p(a+b) &\equiv p(a) + p(b) + h(a,b) && \pmod{K_0}\ , \\ p(at) &\equiv t^\sigma p(a) t && \pmod{K_0}\ , \end{aligned}$$

for all $a, b \in V_0$ and all $t \in K$.

Again following [52], we define a *pseudo-quadratic space* as a quintuple (K, K_0, σ, V_0, p) such that (K, K_0, σ) is an involutory set, V_0 is a right vector space over K and p is a pseudo-quadratic form on V_0 with respect to K_0 and σ. A pseudo-quadratic space (K, K_0, σ, V_0, p) is called *anisotropic* if $p(a) \in K_0$ only for $a = 0$.

Let (K, K_0, σ, V_0, p) be an arbitrary anisotropic pseudo-quadratic space with corresponding skew-hermitian form h. We define a group $(T, \boxplus)$ as

$$T := \left\{(a,t) \in V_0 \times K \mid p(a) - t \in K_0\right\}\ ,$$

where the group action is given by

$$(a,t) \boxplus (b,s) := \left(a+b, t+s+h(b,a)\right)\ ,$$

for all $(a,t), (b,s) \in T$. One can check that T is indeed a group with neutral element $(0,0)$, and with the inverse given by $\boxminus(a,t) = (-a, -t + h(a,a))$, for all $(a,t) \in T$.

2 Quadrangular Systems

Very recently, the classification of Moufang polygons has been completed by J. Tits and R. Weiss in [52]. It was first shown by J. Tits [46, 48] that Moufang n-gons exist for $n \in \{3, 4, 6, 8\}$ only; see also [54]. For $n \in \{3, 6, 8\}$, the proof is divided into two parts, namely (A) it is shown that a Moufang n-gon can be parametrized by a certain algebraic structure, and (B) these algebraic structures are classified.

More precisely, it was already shown in 1933 (but in a slightly different form[1]; see [4] or [21]) by R. Moufang (see [33]) that all Moufang triangles can be described by an alternative division ring, a notion which had been introduced by M. Zorn (see [55]). These alternative division rings were classified by R. Bruck and E. Kleinfeld in 1951; see [10]. The Moufang hexagons are described by unital quadratic Jordan division algebras of degree three, also known as anisotropic cubic norm structures (see [45]). These structures have been classified in its full generality in 1986 by H. Petersson and M. Racine [36, 37], whose proof is built on earlier work by A. Albert [2], F.D. Jacobson and N. Jacobson [23], N. Jacobson [24, 25] and K. McCrimmon [31, 32]. The Moufang octagons, finally, can be described by a so-called octagonal system, as was shown by J. Tits in 1983 (see [49]); since these systems have a very simple description, there is no need for part (B) in this case.

The classification of Moufang quadrangles (n=4) in [52] is not organized in this way due to the absence of a suitable algebraic structure. Instead, there are six different parameter systems, and even then, the division of the proof into parts (A) and (B) is missing in the two cases which lead to the exceptional quadrangles. Surprisingly, one of these classes, namely the exceptional quadrangles of type F_4, had only been discovered by R. Weiss

[1] See the long footnote on page 176 of [52] for the full story.

during the classification process, more than 20 years after Tits's conjecture in [45] that the Moufang polygons could be classified; see also [34].

The goal of this chapter is to present a uniform algebraic structure for Moufang quadrangles. These "quadrangular systems", as we will call them, reveal some of the structure of Moufang quadrangles which is hard to see without them. For example, we have successfully used them to answer a basic question about the automorphism group of the Moufang quadrangles of type F_4 left open in (37.38) of [52]; see chapter 3. Moreover, it is possible to classify these structures without referring back to the original Moufang quadrangles from which they arise, thereby providing a new proof for the classification of Moufang quadrangles, which does consist of the division into parts (A) and (B).

We start by giving the (ad hoc) definition of the quadrangular systems. Considering the background of the Moufang quadrangles, it should not be too surprising that we need a large number of axioms to describe these systems. In the next section, we give some elementary properties of these systems. In section 2.3, we explain how to construct a Moufang quadrangle starting from an arbitrary quadrangular system. In section 2.4, we show that every Moufang quadrangle arises in this way. After a couple of remarks in section 2.5, we present a list of 6 examples of quadrangular systems, which corresponds to the 6 different classes of Moufang quadrangles as described in [52]. Section 2.7, which makes up the largest part of this chapter, is devoted to the classification of the quadrangular systems. We conclude with a section in which we restate the axiom system for abelian quadrangular systems and for some specific subclasses of those.

2.1 Definition

Consider an abelian group $(V, +)$ and a (possibly non-abelian) group $(W, \boxplus)$. The inverse of an element $w \in W$ will be denoted by $\boxminus w$, and by $w_1 \boxminus w_2$, we mean $w_1 \boxplus (\boxminus w_2)$. Suppose that there is a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W, both of which will be denoted by $\cdot$ or simply by juxtaposition, i.e. $\tau_V(v, w) = vw = v \cdot w$ and $\tau_W(w, v) = wv = w \cdot v$ for all $v \in V$ and all $w \in W$. Consider a map F from $V \times V$ to W and a map H from $W \times W$ to V, both of which are "bi-additive" in the sense that

$$F(v_1 + v_2, v) = F(v_1, v) \boxplus F(v_2, v)\,;$$
$$F(v, v_1 + v_2) = F(v, v_1) \boxplus F(v, v_2)\,;$$

$$H(w_1 \boxplus w_2, w) = H(w_1, w) + H(w_2, w)\,;$$
$$H(w, w_1 \boxplus w_2) = H(w, w_1) + H(w, w_2)\,;$$

for all $v, v_1, v_2 \in V$ and all $w, w_1, w_2 \in W$. Suppose furthermore that there exists a fixed element $\epsilon \in V^*$ and a fixed element $\delta \in W^*$, and suppose that, for each $v \in V^*$, there exists an element $v^{-1} \in V^*$, and for each $w \in W^*$, there exists an element $\kappa(w) \in W^*$, such that, for all $w, w_1, w_2 \in W$ and all $v, v_1, v_2 \in V$, the following axioms are satisfied. We define

$$\begin{aligned} \overline{v} &:= \epsilon F(\epsilon, v) - v \\ \mathrm{Rad}(F) &:= \{v \in V \mid F(v, V) = 0\} \\ \mathrm{Rad}(H) &:= \{w \in W \mid H(w, W) = 0\} \\ \mathrm{Im}(F) &:= F(V, V) \\ \mathrm{Im}(H) &:= H(W, W) \end{aligned}$$

(**Q**$_1$) $w\epsilon = w$.

(**Q**$_2$) $v\delta = v$.

(**Q**$_3$) $(w_1 \boxplus w_2)v = w_1 v \boxplus w_2 v$.

(**Q**$_4$) $(v_1 + v_2)w = v_1 w + v_2 w$.

(**Q**$_5$) $w(-\epsilon) \cdot v = w(-v)$.

(**Q**$_6$) $v \cdot w(-\epsilon) = vw$.

(**Q**$_7$) $\mathrm{Im}(F) \subseteq \mathrm{Rad}(H)$.

(**Q**$_8$) $[w_1, w_2 v]_{\boxplus} = F(H(w_2, w_1), v)$.

(**Q**$_9$) $\delta \in \mathrm{Rad}(H)$.

(**Q**$_{10}$) If $\mathrm{Rad}(F) \neq 0$, then $\epsilon \in \mathrm{Rad}(F)$.

(**Q**$_{11}$) $w(v_1 + v_2) = wv_1 \boxplus wv_2 \boxplus F(v_2 w, v_1)$.

(**Q**$_{12}$) $v(w_1 \boxplus w_2) = vw_1 + vw_2 + H(w_2, w_1 v)$.

(**Q**$_{13}$) $(v^{-1})^{-1} = v$ (if $v \neq 0$).

(**Q**$_{14}$) $\kappa(\boxminus\kappa(\boxminus w)) = w(-\epsilon)$ (if $w \neq 0$).

(**Q**$_{15}$) $wv \cdot v^{-1} = w$ (if $v \neq 0$).

(**Q**$_{16}$) $v^{-1} \cdot wv = -\overline{v(\boxminus w)}$ (if $v \neq 0$).

(**Q**$_{17}$) $F(v_1^{-1}, \overline{v_2})v_1 = F(v_1, v_2)$ (if $v_1 \neq 0$).

(**Q**$_{18}$) $v\kappa(w) \cdot (\boxminus w) = -v$ (if $w \neq 0$).

(**Q**$_{19}$) $w \cdot v\kappa(w) = \kappa(w)v$ (if $w \neq 0$).

(**Q**$_{20}$) $H(\kappa(w_1), w_2)w_1 = H(w_1, w_2)$ (if $w_1 \neq 0$).

Then we call the system $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ a *quadrangular system*. Note that we omit the maps F and H in our notation, as well as the maps $v \mapsto v^{-1}$ and $w \mapsto \kappa(w)$. The reason is that they are uniquely determined by V, W, τ_V, τ_W, ϵ and δ; see Theorem 2.2.7.

Remark 2.1.1. We will sometimes think about the maps τ_V from $V \times W$ to V and τ_W from $W \times V$ to W as "actions", since it will turn out that, for every $w \in W^*$, the map from V to itself which maps v to vw for every $v \in V$ is an automorphism of V; similarly, for every $v \in V^*$, the map from W to itself which maps w to wv for every $w \in W$ is an automorphism of W; see Theorem 2.2.6. Note, however, that these maps are no group actions in the proper sense of the word, since $v(w_1 \boxplus w_2) \neq vw_1 \cdot w_2$ and $w(v_1 + v_2) \neq wv_1 \cdot v_2$ in general.

Remark 2.1.2. In writing down these axioms, we used the convention that the maps which are denoted by juxtaposition preceed those which are denoted by "$\cdot$". Note, however, that there is no danger of confusion, since we have not defined a multiplication on V or on W. Hence we will often write wvv^{-1} instead of $wv \cdot v^{-1}$, for example.

We will show in Theorem 2.2.8 below that the following two identities are satisfied for every quadrangular system, for all $v_1, v_2 \in V$ and all $w_1, w_2 \in W$.

($\mathbf{Q}_{21}$) $F(v_1, v_2) = F(v_2, v_1)$.

($\mathbf{Q}_{22}$) $H(w_1, w_2) = -\overline{H(w_2, w_1)}$.

Remark 2.1.3. These two identities show that, in some sense, F is a symmetric form and H is a skew-hermitian form. Note, however, that V and W are *not* vector spaces in general.

Moreover, we will show in Theorem 2.5.1 that the following four identities are satisfied for every quadrangular system, for all $v, c \in V$ and all $w, z \in W$. We first introduce the notion of a *reflection*, which is a direct generalization of the classical notion of a reflection in a quadratic space as defined on page 11:

$$\begin{aligned} \pi_v(c) &:= c - vF(v^{-1}, \overline{c}) && \text{(if } v \neq 0) \\ \Pi_w(z) &:= z \boxplus w(-H(\kappa(w), z)) && \text{(if } w \neq 0)\ . \end{aligned}$$

Then

($\mathbf{Q}_{23}$) $v \cdot \Pi_w(z) = -\overline{\overline{\overline{\overline{v}(\boxminus w)}z}\kappa(w)}$ (if $w \neq 0$).

($\mathbf{Q}_{24}$) $w \cdot \overline{\pi_v(\epsilon)^{-1}} \cdot \overline{\pi_v(c)} = wvcv^{-1}$ (if $v \neq 0$).

($\mathbf{Q}_{25}$) $\pi_v(\overline{c \cdot \delta v})w = \pi_v(\overline{c \cdot wv})$ (if $v \neq 0$).

($\mathbf{Q}_{26}$) $\Pi_{\boxminus z}(w \cdot \epsilon z)v = \Pi_{\boxminus z}(w \cdot vz)$ (if $w \neq 0$).

Let $\Omega := (V, W, \tau_V, \tau_W, \epsilon, \delta)$ and $\Omega' := (V', W', \tau_{V'}, \tau_{W'}, \epsilon', \delta')$ be two quadrangular systems. We say that (ϕ, ψ) is a *morphism* from Ω to Ω' if and only if ϕ is a morphism from V to V' and ψ is a morphism from W to W' such that $\phi(\epsilon) = \epsilon'$, $\psi(\delta) = \delta'$, $\phi(vw) = \phi(v)\psi(w)$ and $\psi(wv) = \psi(w)\phi(v)$, for all $v \in V$ and all $w \in W$.

A morphism (ϕ, ψ) is called an *monomorphism* (respectively *epimorphism*, *isomorphism*) if and only if both ϕ and ψ are monomorphisms (respectively epimorphisms, isomorphisms). We call Ω and Ω' *isomorphic* if and only if there exists an isomorphism (ϕ, ψ) from Ω to Ω'.

2.2 Some Identities

We will now prove some identities which we will use in the construction of the Moufang quadrangles in section 2.3, and which will also be used in the classification of quadrangular systems.

Definition 2.2.1. For each $w \in W^*$, we define $\lambda(w) := \boxminus\kappa(\boxminus w)$. Using this definition, ($\mathbf{Q}_{14}$) can be rephrased as $\kappa(\lambda(w)) = w(-\epsilon)$.

Lemma 2.2.2. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then, for all $w \in W^*$ and all $v \in V$, we have that*

(i) $vw\kappa(\boxminus w) = -v$*;*

(ii) $\kappa(w)(v(\boxminus w)) = w(-v)$.

Proof. If we plug in $\lambda(w)$ for w in ($\mathbf{Q}_{18}$), then it follows from ($\mathbf{Q}_{14}$) that $v(w(-\epsilon))(\boxminus\lambda(w)) = -v$, and by ($\mathbf{Q}_6$) and the definition of λ, this is equivalent to $vw\kappa(\boxminus w) = -v$, which proves (i).

If we plug in $\lambda(w)$ for w in ($\mathbf{Q}_{19}$), then we get, again by ($\mathbf{Q}_{14}$), that $\lambda(w)(v \cdot w(-\epsilon)) = w(-\epsilon)v$. By ($\mathbf{Q}_6$), ($\mathbf{Q}_5$) and the definition of λ, this is equivalent to $\boxminus\kappa(\boxminus w)(vw) = w(-v)$. Replacing w by $\boxminus w$ now yields (ii). $\square$

Lemma 2.2.3. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then the following holds, for all $w \in W$ and all $v \in V$:*

(i) $wv = 0 \iff w = 0$ *or* $v = 0$*;*

(ii) $vw = 0 \iff v = 0$ *or* $w = 0$.

Proof. We will only prove statement (i); because of Lemma 2.2.2(i), the proof of (ii) is completely similar. By choosing $v_1 = v_2 = 0$ in ($\mathbf{Q}_{11}$), we get $w0 = w0 \boxplus w0$, from which it follows that $w0 = 0$. Similarly, it follows from ($\mathbf{Q}_3$) that $0v = 0$.

On the other hand, suppose that $wv = 0$. If $v \neq 0$, then it follows from ($\mathbf{Q}_{15}$) that $w = wvv^{-1} = 0v^{-1} = 0$. □

Lemma 2.2.4. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then, for all $w \in W$ and all $v \in V$, we have :*

(i) $(\boxminus w)v = \boxminus(wv)$ *;*

(ii) $(-v)w = -(vw)$ *.*

It follows that the notations $\boxminus wv$ and $-vw$ are unambiguous.

Proof. By putting $w_1 = w$ and $w_2 = \boxminus w$ in ($\mathbf{Q}_3$), we get $0v = wv \boxplus (\boxminus w)v$, from which it follows that $(\boxminus w)v = \boxminus(wv)$. Similarly, (ii) follows from ($\mathbf{Q}_4$). □

Lemma 2.2.5. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then the following holds, for all $w \in W$ and all $v \in V$:*

(i) $w_1 v = w_2 v \iff w_1 = w_2$ *or* $v = 0$*;*

(ii) $v_1 w = v_2 w \iff v_1 = v_2$ *or* $w = 0$*.*

Proof. By ($\mathbf{Q}_3$) and Lemma 2.2.4(i), we have $(w_1 \boxminus w_2)v = w_1 v \boxminus w_2 v$, and so (i) is an immediate consequence of Lemma 2.2.3(i). Similarly, (ii) follows from Lemma 2.2.3(ii). □

Theorem 2.2.6. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then*

(i) *for every $w \in W^*$, the map from V to itself which maps v to vw for every $v \in V$ is an automorphism of V;*

(ii) *for every $v \in V^*$, the map from W to itself which maps w to wv for every $w \in W$ is an automorphism of W.*

Proof. We will only show (i), the proof of (ii) being completely similar. So let $w \in W^*$ be arbitrary, and let α be the map from V to itself which maps v to vw for every $v \in V$. By Lemma 2.2.3(ii), we have that $\alpha(0) = 0$, and it follows from ($\mathbf{Q}_4$) that $\alpha(v_1 + v_2) = \alpha(v_1) + \alpha(v_2)$ for all $v_1, v_2 \in V$, so α is a group morphism. Since $w \neq 0$, it follows from Lemma 2.2.5(ii) that α is injective. Finally, it follows from ($\mathbf{Q}_{18}$) that $\alpha\big(-v\kappa(\boxminus w)\big) = v$ for all $v \in V$, hence α is surjective as well, and we are done. □

Theorem 2.2.7. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then the maps F and H and the maps $v \mapsto v^{-1}$ and $w \mapsto \kappa(w)$ are uniquely determined.*

Proof. By $(\mathbf{Q}_{11})$, $F(v_2, v_1) = \boxminus\delta v_2 \boxminus \delta v_1 \boxplus \delta(v_1 + v_2)$, so F is uniquely determined. Note that this implies that the map $v \mapsto \overline{v}$ is uniquely determined as well. By $(\mathbf{Q}_{12})$, $H(w_2, w_1) = -\epsilon w_2 - \epsilon w_1 + \epsilon(w_1 + w_2)$, so H is uniquely determined. Suppose that v^* were another "inverse" of v. Then it would follow from $(\mathbf{Q}_{16})$ that $v^*(wv) = v^{-1}(wv)$, but then Lemma 2.2.5 would imply that $v^* = v^{-1}$ after all. Similarly, it follows from Lemma 2.2.2(ii) that the map κ is uniquely determined. □

Theorem 2.2.8. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then the identities $(\mathbf{Q}_{21})$ and $(\mathbf{Q}_{22})$ are satisfied for all $v_1, v_2 \in V$ and all $w_1, w_2 \in W$.*

Proof. We will first show that $(\mathbf{Q}_{21})$ follows from $(\mathbf{Q}_8)$, $(\mathbf{Q}_9)$ and $(\mathbf{Q}_{11})$. Since V is abelian, $\delta(v_1 + v_2) = \delta(v_2 + v_1)$, and hence, by $(\mathbf{Q}_{11})$, we have that

$$\delta v_1 \boxplus \delta v_2 \boxplus F(v_2, v_1) = \delta v_2 \boxplus \delta v_1 \boxplus F(v_1, v_2) ,$$

for all $v_1, v_2 \in V$. In order to show $(\mathbf{Q}_{21})$, it thus suffices to show that δv_1 and δv_2 commute for all $v_1, v_2 \in V$. By $(\mathbf{Q}_8)$ and $(\mathbf{Q}_9)$,

$$[\delta v_1, \delta v_2]_{\boxplus} = F(H(\delta, \delta v_1), v_2) = 0$$

for all $v_1, v_2 \in V$, and hence $(\mathbf{Q}_{21})$ holds.

Similarly, we will show that $(\mathbf{Q}_{22})$ follows from $(\mathbf{Q}_{12})$, $(\mathbf{Q}_{15})$ and $(\mathbf{Q}_{16})$. By substituting $\boxminus w$ for w and ϵ for v in $(\mathbf{Q}_{16})$, we have that $\overline{\epsilon w} = -\epsilon^{-1}(\boxminus w)$ for all $w \in W$. Moreover, by $(\mathbf{Q}_{15})$, $w\epsilon^{-1} = w$ for all $w \in W$. By $(\mathbf{Q}_{12})$ and the fact that H is additive in both variables, we thus have that

$$\begin{aligned} \overline{H(w_2, w_1)} &= \overline{\epsilon(w_1 \boxplus w_2)} - \overline{\epsilon w_1} - \overline{\epsilon w_2} \\ &= -\epsilon^{-1}(\boxminus w_2 \boxminus w_1) + \epsilon^{-1}(\boxminus w_1) + \epsilon^{-1}(\boxminus w_2) \\ &= -H(\boxminus w_1, \boxminus w_2 \epsilon^{-1}) \\ &= -H(w_1, w_2) \end{aligned}$$

for all $w_1, w_2 \in W$, hence $(\mathbf{Q}_{22})$ holds. □

Lemma 2.2.9. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then the map $v \mapsto \overline{v}$ is additive. In particular, we have that $\overline{-v} = -\overline{v}$ for all $v \in V$. Moreover, for all $c \in V^*$, the map π_c is additive. In particular, we have that $\pi_c(-v) = -\pi_c(v)$ for all $v \in V$.*

Proof. It follows from ($\mathbf{Q}_7$) that $H(F(\epsilon, v_2), F(\epsilon, v_1)) = 0$, for all $v_1, v_2 \in V$. Hence

$$\begin{aligned} \epsilon F(\epsilon, v_1 + v_2) &= \epsilon(F(\epsilon, v_1) \boxplus F(\epsilon, v_2)) \\ &= \epsilon F(\epsilon, v_1) + \epsilon F(\epsilon, v_2) \ , \end{aligned}$$

by ($\mathbf{Q}_{12}$). Since $\overline{v} = \epsilon F(\epsilon, v) - v$, it follows from this that the map $v \mapsto \overline{v}$ is additive. Similarly, it follows from ($\mathbf{Q}_7$) that $H(F(c^{-1}, \overline{v_2}), F(c^{-1}, \overline{v_1})c) = 0$, for all $c \in V^*$ and all $v_1, v_2 \in V$. Since the map $v \mapsto \overline{v}$ is additive, it now follows, again by ($\mathbf{Q}_{12}$), that

$$\begin{aligned} \pi_c(v_1 + v_2) &= (v_1 + v_2) - cF(c^{-1}, \overline{v_1 + v_2}) \\ &= v_1 + v_2 - c(F(c^{-1}, \overline{v_1}) \boxplus F(c^{-1}, \overline{v_2})) \\ &= v_1 - cF(c^{-1}, \overline{v_1}) + v_2 - cF(c^{-1}, \overline{v_2}) \\ &= \pi_c(v_1) + \pi_c(v_2) \ , \end{aligned}$$

which is what we wanted to show. □

Lemma 2.2.10. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then, for all $w \in W^*$ and all $v \in V^*$, we have*

(i) $(-v)^{-1} = -(v^{-1})$;

(ii) $\kappa(\boxminus w) = \boxminus\lambda(w)$.

Proof. If we replace w by δ in ($\mathbf{Q}_{16}$), we have that $-(v^{-1})(\delta v) = \overline{v(\boxminus\delta)}$. If we replace w by $\delta(-\epsilon)$ in the same identity ($\mathbf{Q}_{16}$), then we get, by ($\mathbf{Q}_5$) and ($\mathbf{Q}_6$) that $v^{-1}(\delta(-v)) = -\overline{v(\boxminus\delta)}$. If we replace v by $-v$ in this identity, then we get, using the fact that $\overline{-v} = -\overline{v}$, that $(-v)^{-1}(\delta v) = \overline{v(\boxminus\delta)}$. It follows that $(-v)^{-1}(\delta v) = -(v^{-1})(\delta v)$. Since δv is non-zero, this implies, by Lemma 2.2.5(i), that $(-v)^{-1} = -(v^{-1})$, which proves (i). Identity (ii) follows immediately from the definition of λ. □

Lemma 2.2.11. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then, for all $w \in W$ and all $v \in V$, we have*

$$wv(-\epsilon) = w(-v) \ .$$

Proof. Note that this identity is trivial if $v = 0$, so assume $v \neq 0$. By ($\mathbf{Q}_{15}$) and Lemma 2.2.10(i), we have that $wvv^{-1} = w(-v)(-v^{-1})$. It follows, by ($\mathbf{Q}_5$), that $wv(-\epsilon)(-v^{-1}) = w(-v)(-v^{-1})$. By Lemma 2.2.5(i), this implies that $wv(-\epsilon) = w(-v)$, for all $w \in W$ and all $v \in V$. □

Lemma 2.2.12. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then, for all $v \in V$, we have*

$$\overline{\overline{v}} = v\,.$$

Proof. Assume $v \neq 0$. By replacing w by $\boxminus\delta$ in ($\mathbf{Q}_{16}$), we see that $\overline{v} = -v^{-1}(\boxminus\delta v)$. If, on the other hand, we replace v by v^{-1} and w by δv in this same identity ($\mathbf{Q}_{16}$), then we get $v(\delta v v^{-1}) = -\overline{v^{-1}(\boxminus\delta v)}$. Combining those two equalities gives us the required identity $\overline{\overline{v}} = v$, since $v(\delta v v^{-1})$ is equal to v because of ($\mathbf{Q}_{15}$) and ($\mathbf{Q}_2$). ☐

Lemma 2.2.13. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then, for all $w \in W$ and all $v \in V$, we have*

(i) $w(-v) = F(vw, v) \boxminus wv$;

(ii) $v(\boxminus w) = H(w, wv) - vw$.

Proof. If we put $v_1 = -v$ and $v_2 = v$ in ($\mathbf{Q}_{11}$), then we get that $w0 = w(-v) \boxplus wv \boxplus F(vw, -v)$. Since F is additive in both variables, this is equivalent to $w(-v) = F(vw, v) \boxminus wv$, which proves (i). Similarly, (ii) follows from ($\mathbf{Q}_{12}$). ☐

Lemma 2.2.14. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then, for all $v \in V^*$, we have*

(i) $\kappa(\delta v) = \delta(\overline{v})^{-1}$;

(ii) $\overline{v^{-1}} = (\overline{v})^{-1}$.

Proof. If we substitute δv for w and $-v^{-1}$ for v in Lemma 2.2.2(ii), then we get that

$$\kappa(\delta v) \cdot (-v^{-1}(\boxminus\delta v)) = \delta v v^{-1}\,,$$

and hence, by ($\mathbf{Q}_{16}$) and ($\mathbf{Q}_{15}$),

$$\kappa(\delta v) \cdot \overline{v} = \delta\,.$$

By ($\mathbf{Q}_{15}$), it thus follows that $\kappa(\delta v) = \delta(\overline{v})^{-1}$, which shows (i). Note that it follows from Lemma 2.2.13(ii) that $v(\boxminus\delta) = -v$ for all $v \in V$, since $\delta \in \mathrm{Rad}(H)$ by ($\mathbf{Q}_9$). By Lemma 2.2.2(i) with $\boxminus\delta\overline{v}$ in place of w, ($\mathbf{Q}_{16}$) with $\overline{v}$ in place of v and $\boxminus\delta$ in place of w, Lemma 2.2.12, and (i) with $\overline{v}$ in place of v, we now have that

$$\begin{aligned}(\overline{v})^{-1} &= -(\overline{v})^{-1} \cdot (\boxminus\delta\overline{v}) \cdot \kappa(\delta\overline{v}) \\ &= v \cdot \delta v^{-1} = -\overline{v^{-1}(\boxminus\delta)} = \overline{v^{-1}}\,,\end{aligned}$$

which shows (ii). ☐

Lemma 2.2.15. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then, for all $v_1, v_2 \in V$ and all $w_1, w_2 \in W$, we have that*

(i) $F(v_1, v_2)(-\epsilon) = F(v_1, v_2)$*;*

(ii) $H(\kappa(w_1(-\epsilon)), w_2) = -H(\kappa(w_1), w_2)$*.*

Proof. If we substitute $\overline{v_2}$ for v_2 in ($\mathbf{Q}_{17}$), then we get, using ($\mathbf{Q}_{21}$), that $F(v_2, v_1^{-1})v_1 = F(v_1, \overline{v_2})$. Replacing v_1 by $-v_1$ in this last identity and applying Lemma 2.2.10(i) yields $F(v_2, v_1^{-1})(-v_1) = F(v_1, \overline{v_2})$. Thus, by ($\mathbf{Q}_5$), $F(v_2, v_1^{-1})(-\epsilon)v_1 = F(v_2, v_1^{-1})v_1$, and it follows from Lemma 2.2.5(i) that $F(v_2, v_1^{-1})(-\epsilon) = F(v_2, v_1^{-1})$. Replacing v_1 by v_1^{-1} and using ($\mathbf{Q}_{13}$) completes the proof of (i).

The proof of (ii) is similar. If we substitute $w_1(-\epsilon)$ for w_1 in ($\mathbf{Q}_{20}$), then we get that $H(\kappa(w_1(-\epsilon)), w_2) \cdot w_1(-\epsilon) = H(w_1(-\epsilon), w_2)$. On the other hand, since $\mathrm{Im}(F) \subseteq \mathrm{Rad}(H)$ by ($\mathbf{Q}_7$), it follows from Lemma 2.2.13(i) that $H(w_1(-\epsilon), w_2) = H(\boxminus w_1, w_2) = -H(w_1, w_2)$. Hence

$$H(\kappa(w_1(-\epsilon)), w_2) \cdot w_1(-\epsilon) = -H(w_1, w_2)\ ,$$

and it follows from ($\mathbf{Q}_6$) and ($\mathbf{Q}_{20}$) that

$$H(\kappa(w_1(-\epsilon)), w_2) \cdot w_1 = -H(\kappa(w_1), w_2) \cdot w_1\ .$$

It now follows from Lemma 2.2.5(ii) that (ii) holds. □

Lemma 2.2.16. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then, for all $w \in W$ and all $v \in V$, we have*

(i) $F(\overline{vw}, v^{-1}) \boxminus w(-v)v^{-1} = w$ *(if $v \neq 0$);*

(ii) $H(w, \kappa(w)(-v)) + v\kappa(w)w = v$ *(if $w \neq 0$).*

Proof. Putting $v_1 = v$ and $v_2 = vw$ in ($\mathbf{Q}_{17}$), and using ($\mathbf{Q}_{21}$), yields $F(\overline{vw}, v^{-1})v = F(vw, v)$, from which it follows, by ($\mathbf{Q}_{15}$), that $F(\overline{vw}, v^{-1}) = F(vw, v)v^{-1}$. It follows from Lemma 2.2.13(i) and from ($\mathbf{Q}_3$) that

$$\begin{aligned} w(-v)v^{-1} &= (F(vw, v) \boxminus wv)v^{-1} \\ &= F(vw, v)v^{-1} \boxminus wvv^{-1} \\ &= F(\overline{vw}, v^{-1}) \boxminus w\ , \end{aligned}$$

from which (i) follows, since $\mathrm{Im}(F) \subseteq Z(W)$ by ($\mathbf{Q}_7$) and ($\mathbf{Q}_8$).

If we plug in $v\kappa(w)$ for v in Lemma 2.2.13(ii), we get

$$v\kappa(w)(\boxminus w) = H(w, w \cdot v\kappa(w)) - v\kappa(w)w\ ,$$

and applying ($\mathbf{Q}_{18}$) and ($\mathbf{Q}_{19}$) yields $-v = H(w, \kappa(w)v) - v\kappa(w)w$. Replacing v by $-v$ gives us the required identity (ii). □

Theorem 2.2.17. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then, for all $w_1 \in W^*$, $w_2 \in W$, $v_1 \in V^*$ and $v_2 \in V$, we have*

(i) $F(2v_2 - \overline{v_1 F(v_2, v_1^{-1})}, v_1^{-1}) = 0$;

(ii) $H(\kappa(w_1) \boxplus \lambda(w_1), w_2) + H(\lambda(w_1), w_1(-H(\kappa(w_1), w_2))) = 0$.

Proof. By ($\mathbf{Q}_5$), ($\mathbf{Q}_{15}$) and Lemma 2.2.15(i), we have

$$\begin{aligned} F(v_2, v_1^{-1})(-v_1)v_1^{-1} &= F(v_2, v_1^{-1})(-\epsilon)v_1 v_1^{-1} \\ &= F(v_2, v_1^{-1}) . \end{aligned}$$

If we put $v = v_1$ and $w = F(v_2, v_1^{-1})$ in Lemma 2.2.16(i), we get

$$\begin{aligned} F(\overline{v_1 F(v_2, v_1^{-1})}, v_1^{-1}) &= F(v_2, v_1^{-1}) \boxplus F(v_2, v_1^{-1})(-v_1)v_1^{-1} \\ &= F(v_2, v_1^{-1}) \boxplus F(v_2, v_1^{-1}) \\ &= F(2v_2, v_1^{-1}) , \end{aligned}$$

from which (i) follows.

To prove (ii), we first observe that, by ($\mathbf{Q}_7$), it follows from Lemma 2.2.13(i) that

$$\begin{aligned} H(w_1(-v), w_2) &= H(F(vw_1, v) \boxminus w_1 v, w_2) \\ &= -H(w_1 v, w_2) , \end{aligned}$$

for all $w_1, w_2 \in W$ and all $v \in V$. We also observe that

$$\kappa(\lambda(w))(-v) = w(-\epsilon)(-v) = wv$$

because of ($\mathbf{Q}_{14}$) and ($\mathbf{Q}_5$), and that

$$v\kappa(\lambda(w)) = v(w(-\epsilon)) = vw$$

because of ($\mathbf{Q}_{14}$) and ($\mathbf{Q}_6$), for all $w \in W^*$ and all $v \in V$. If we substitute $\lambda(w_1)$ for w and $-H(\kappa(w_1), w_2)$ for v in Lemma 2.2.16(ii), then we get, using these remarks, that

$$\begin{aligned} &H(\lambda(w_1), w_1(-H(\kappa(w_1), w_2))) \\ &\qquad = -H(\kappa(w_1), w_2) + H(\kappa(w_1), w_2)w_1\lambda(w_1) \\ &\qquad = -H(\kappa(w_1), w_2) + H(w_1, w_2)\lambda(w_1) \\ &\qquad = -H(\kappa(w_1), w_2) - H(w_1(-\epsilon), w_2)\lambda(w_1) \\ &\qquad = -H(\kappa(w_1), w_2) - H(\kappa(\lambda(w_1)), w_2)\lambda(w_1) \\ &\qquad = -H(\kappa(w_1), w_2) - H(\lambda(w_1), w_2) \\ &\qquad = -H(\kappa(w_1) \boxplus \lambda(w_1), w_2) , \end{aligned}$$

where we have used identity ($\mathbf{Q}_{20}$) twice. This completes the proof of (ii). □

Lemma 2.2.18. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then, for all $v \in V^*$, $c \in V$, $w \in W^*$ and $z \in W$, we have that*

(i) $\pi_v(c) = c - vF(v^{-1}, \overline{c}) = c - \overline{v^{-1}F(v,c)}$;

(ii) $\Pi_w(z) = z \boxplus w(-H(\kappa(w), z)) = z \boxplus \lambda(w)H(w,z)$.

Proof. By $(\mathbf{Q}_{17})$, $(\mathbf{Q}_{15})$ and $(\mathbf{Q}_{16})$, we have that

$$\begin{aligned} vF(v^{-1}, \overline{c}) &= v \cdot F(v,c)v^{-1} \\ &= -\overline{v^{-1}(\boxminus F(v,c))} \; . \end{aligned}$$

Since $\mathrm{Im}(F) \subseteq \mathrm{Rad}(H)$ by $(\mathbf{Q}_7)$, it follows from Lemma 2.2.13(ii) that $v^{-1}(\boxminus F(v,c)) = -v^{-1}F(v,c)$, and hence

$$vF(v^{-1}, \overline{c}) = \overline{v^{-1}F(v,c)} \; ,$$

which shows (i).

By $(\mathbf{Q}_{20})$, Lemma 2.2.2(i) and $(\mathbf{Q}_{19})$, we have that

$$\begin{aligned} w(-H(\kappa(w), z)) &= w \cdot (H(w,z)\kappa(\boxminus w)) \\ &= \boxminus \kappa(\boxminus w)H(w,z) \\ &= \lambda(w)H(w,z) \; , \end{aligned}$$

which shows (ii). ☐

In the sequel, we will use both expressions as definitions of π_v and Π_w, without explicitly referring to this lemma.

Lemma 2.2.19. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system, and let $w \in \mathrm{Rad}(H)$ and $v \in V$. Then $wv \in \mathrm{Rad}(H)$ as well.*

Proof. By $(\mathbf{Q}_8)$, $[w, w_2]_\boxplus = 0$ for all $w_2 \in W$, hence $v(w \boxplus w_2) = v(w_2 \boxplus w)$. It follows from $(\mathbf{Q}_{12})$ that $H(w_2, wv) = H(w, w_2 v) = 0$ for all $w_2 \in W$, since $w \in \mathrm{Rad}(H)$. By $(\mathbf{Q}_{22})$, this implies that $H(wv, w_2) = 0$ for all $w_2 \in W$, hence $wv \in \mathrm{Rad}(H)$. ☐

Lemma 2.2.20. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then we have that $vw = -\overline{\overline{v}(\delta\overline{v^{-1}})(\boxminus wv)}$ for all $v \in V^*$ and all $w \in W$.*

Proof. It follows from $(\mathbf{Q}_{16})$ that $\overline{vw} = -v^{-1}(\boxminus wv)$ for all $v \in V^*$ and all $w \in W$. In particular, we have that $\overline{v} = -v^{-1}(\boxminus \delta v)$ for all $v \in V^*$, and hence that $\overline{v}\kappa(\delta v) = v^{-1}$ by Lemma 2.2.2(i). If we substitute this expression for v^{-1} in the first identity, then we get that $\overline{vw} = -\overline{v}\kappa(\delta v)(\boxminus wv)$ for all $v \in V^*$ and all $w \in W$. The result follows, since $\kappa(\delta v) = \delta\overline{v^{-1}}$ for all $v \in V^*$ by Lemma 2.2.14(i) and Lemma 2.2.14(ii). ☐

Lemma 2.2.21. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then $\pi_v(vw) = v(\boxminus w)$ for all $v \in V^*$ and all $w \in W$.*

Proof. Let $v \in V^*$ and $w \in W$ be arbitrary. It follows from ($\mathbf{Q}_5$) and ($\mathbf{Q}_{15}$) that $w(-v)v^{-1} = w(-\epsilon)$. It thus follows from Lemma 2.2.16(i) that $F(\overline{vw}, v^{-1}) = w \boxplus w(-\epsilon)$. Hence, by ($\mathbf{Q}_{12}$), ($\mathbf{Q}_5$), ($\mathbf{Q}_6$) and Lemma 2.2.13(ii),

$$\begin{aligned} vF(\overline{vw}, v^{-1}) &= v(w \boxplus w(-\epsilon)) \\ &= vw + v \cdot w(-\epsilon) + H(w(-\epsilon), wv) \\ &= vw + vw + H(w(-\epsilon), w(-\epsilon)(-v)) \\ &= vw + vw - v(\boxminus w(-\epsilon)) - v(w(-\epsilon)) \\ &= vw + vw - v(\boxminus w) - vw \\ &= vw - v(\boxminus w) \ , \end{aligned}$$

and hence

$$\pi_v(vw) = vw - vF(\overline{vw}, v^{-1}) = v(\boxminus w) \ ,$$

which is what we had to show. □

The following two lemmas generalize some properties of reflections in an ordinary quadratic space.

Lemma 2.2.22. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then*

(i) *$F(v, \pi_v(c)) = F(v, -c)$ for all $v \in V^*$ and all $c \in V$;*

(ii) *$H(\lambda(w), \Pi_w(z)) = -H(\kappa(w), z)$ for all $w \in W^*$ and all $z \in W$.*

Proof. By Lemma 2.2.18(i) and Theorem 2.2.17(i) with v^{-1} in place of v_1 and c in place of v_2, we have that

$$\begin{aligned} F(v, \pi_v(c)) &= F(v, c - \overline{v^{-1}F(v,c)}) \\ &= F(2c - \overline{v^{-1}F(c,v)}, v) \boxminus F(c, v) \\ &= F(v, -c) \ , \end{aligned}$$

which shows (i). By Theorem 2.2.17(ii) with $w_1 = w$ and $w_2 = z$,

$$\begin{aligned} H(\lambda(w), \Pi_w(z)) &= H(\lambda(w), z \boxplus w(-H(\kappa(w), z))) \\ &= H(\kappa(w) \boxplus \lambda(w), z) + H(\lambda(w), w(-H(\kappa(w), z))) \\ &\qquad - H(\kappa(w), z) \\ &= -H(\kappa(w), z) \ , \end{aligned}$$

which shows (ii). □

Lemma 2.2.23. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then*

(i) $\pi_v(\pi_v(c)) = c$ *for all* $v \in V^*$ *and all* $c \in V$ *;*

(ii) $\Pi_{\boxminus w}(\Pi_w(z)) = z$ *for all* $w \in W^*$ *and all* $z \in W$*.*

Proof. By Lemma 2.2.22(i),

$$\begin{aligned}\pi_v(\pi_v(c)) &= \pi_v(c) - \overline{v^{-1}F(v, \pi_v(c))} \\ &= -\pi_v(-c) - \overline{v^{-1}F(v, -c)} \\ &= c + \overline{v^{-1}F(v, -c)} - \overline{v^{-1}F(v, -c)} \\ &= c\ ,\end{aligned}$$

which shows (i). By Lemma 2.2.10(ii) and Lemma 2.2.22(ii),

$$\begin{aligned}\Pi_{\boxminus w}(\Pi_w(z)) &= \Pi_w(z) \boxplus (\boxminus w)(-H(\kappa(\boxminus w), \Pi_w(z))) \\ &= \Pi_w(z) \boxminus wH(\lambda(w), \Pi_w(z)) \\ &= \Pi_w(z) \boxminus w(-H(\kappa(w), z) \\ &= z\ ,\end{aligned}$$

which shows (ii). □

2.3 From Quadrangular Systems To Moufang Quadrangles

We will now describe how we can construct a Moufang quadrangle from a quadrangular system. We will use the method described in section 1.4. Therefore, we will describe 4 groups U_1, U_2, U_3 and U_4, and we will implicitly define the group $U_+ := \langle U_1, U_2, U_3, U_4 \rangle$ by giving the commutator relations between any two of those groups. In order to show that the construction of the graph Ξ out of this sequence $(U_+, U_1, U_2, U_3, U_4)$ will actually result in a Moufang quadrangle, we will follow Theorems 1.4.7 and 1.4.4.

Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Let U_1 and U_3 be two groups isomorphic to W, and let U_2 and U_4 be two groups isomorphic to V. Denote the corresponding isomorphisms by

$$\begin{aligned}x_1 &: W \to U_1 : w \mapsto x_1(w)\ ; \\ x_2 &: V \to U_2 : v \mapsto x_2(v)\ ; \\ x_3 &: W \to U_3 : w \mapsto x_3(w)\ ; \\ x_4 &: V \to U_4 : v \mapsto x_4(v)\ ;\end{aligned}$$

we say that U_1 and U_3 are *parametrized* by W and that U_2 and U_4 are *parametrized* by V. Now, we implicitly define the group $U_+ = U_{[1,4]}$ by the following commutator relations. Note that we continue to use the notation vw or $v \cdot w$ for $\tau_V(v,w)$ and wv or $w \cdot v$ for $\tau_W(w,v)$.

$$\begin{aligned} [x_1(w_1), x_3(w_2)^{-1}] &= x_2(H(w_1,w_2)) \;, \\ [x_2(v_1), x_4(v_2)^{-1}] &= x_3(F(v_1,v_2)) \;, \\ [x_1(w), x_4(v)^{-1}] &= x_2(vw)x_3(wv) \;, \\ [U_i, U_{i+1}] &= 1 \quad \forall i \in \{1,2,3\} \;, \end{aligned} \tag{2.1}$$

for all $w, w_1, w_2 \in W$ and all $v, v_1, v_2 \in V$. We will denote the corresponding graph Ξ by $\mathcal{Q}(\Omega) = \mathcal{Q}(V, W, \tau_V, \tau_W, \epsilon, \delta)$. If we define

$$\begin{aligned} \xi_{13}(x_1(w_1), x_3(w_2)^{-1}) &= x_2(H(w_1,w_2)) \;, \\ \xi_{24}(x_2(v_1), x_4(v_2)^{-1}) &= x_3(F(v_1,v_2)) \;, \\ \xi_{14}(x_1(w), x_4(v)^{-1}) &= x_2(vw)x_3(wv) \;, \end{aligned}$$

then we can rephrase the conditions $\mathcal{A}_k$, $\mathcal{B}_k$ and $\mathcal{C}_k$ as follows.

For all $(i,j) \in \{(1,3),(2,4),(1,4)\}$, the following conditions should hold, for all $a_i, b_i \in U_i$, for all $a_j, b_j \in U_j$, and for all $c \in U_{[i+1,j-1]}$.

$\mathcal{A}_{ij}$. $\quad \xi_{ij}(a_i b_i, a_j^{-1}) = \xi_{ij}(a_i, a_j^{-1})^{b_i} \xi_{ij}(b_i, a_j^{-1})$.

$\mathcal{B}_{ij}$. $\quad \xi_{ij}(a_i, (a_j b_j)^{-1}) = \xi_{ij}(a_i, a_j^{-1}) \xi_{ij}(a_i, b_j^{-1})^{a_j^{-1}}$.

$\mathcal{C}_{ij}$. $\quad c^{\xi_{ij}(a_i, a_j^{-1})} = c^{a_i^{-1} a_j a_i a_j^{-1}}$.

Theorem 2.3.1. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then the corresponding graph $\mathcal{Q}(V, W, \tau_V, \tau_W, \epsilon, \delta)$ satisfies all of the conditions $\mathcal{A}_{ij}$, $\mathcal{B}_{ij}$ and $\mathcal{C}_{ij}$.*

Proof. By plugging in the formulas for the functions ξ_{ij}, we get the following explicit conditions, which must hold for all $v, v', v_1, v_2 \in V$ and all $w, w', w_1, w_2 \in W$.

$\mathcal{A}_{13}$. $\quad x_2(H(w_1 \boxplus w_2, w')) = x_2(H(w_1, w'))^{x_1(w_2)} x_2(H(w_2, w'))\,;$

$\mathcal{A}_{24}$. $\quad x_3(F(v_1 + v_2, v')) = x_3(F(v_1, v'))^{x_2(v_2)} x_3(F(v_2, v'))\,;$

$\mathcal{A}_{14}$. $$\begin{aligned} &x_2(v(w_1 \boxplus w_2)) x_3((w_1 \boxplus w_2)v) \\ &\qquad = (x_2(vw_1)x_3(w_1 v))^{x_1(w_2)} \cdot (x_2(vw_2)x_3(w_2 v))\,; \end{aligned}$$

$\mathcal{B}_{13}$. $\quad x_2(H(w', w_1 \boxplus w_2)) = x_2(H(w', w_1)) x_2(H(w', w_2))^{x_3(\boxminus w_2)}\,;$

$\mathcal{B}_{24}$. $\quad x_3(F(v', v_1 + v_2)) = x_3(F(v', v_1)) x_3(F(v', v_2))^{x_4(-v_2)}\,;$

$\mathcal{B}_{14}$. $\quad x_2((v_1+v_2)w)x_3(w(v_1+v_2))$
$$= (x_2(v_1w)x_3(wv_1)) \cdot (x_2(v_2w)x_3(wv_2))^{x_4(-v_1)} ;$$

$\mathcal{C}_{13}$. $\quad x_2(v)^{x_2(H(w,w'))} = x_2(v)^{x_1(\boxminus w)x_3(w')x_1(w)x_3(\boxminus w')}$;

$\mathcal{C}_{24}$. $\quad x_3(w)^{x_3(F(v,v'))} = x_3(w)^{x_2(-v)x_4(v')x_2(v)x_4(-v')}$;

$\mathcal{C}_{14,2}$. $\quad x_2(v')^{x_2(vw)x_3(wv)} = x_2(v')^{x_1(\boxminus w)x_4(v)x_1(w)x_4(-v)}$;

$\mathcal{C}_{14,3}$. $\quad x_3(w')^{x_2(vw)x_3(wv)} = x_3(w')^{x_1(\boxminus w)x_4(v)x_1(w)x_4(-v)}$.

Note that $[U_1, U_2] = [U_2, U_3] = [U_3, U_4] = 1$; some of the conditions can be simplified by this observation.

Condition ($\mathcal{A}_{13}$) is equivalent to

$$x_2(H(w_1 \boxplus w_2, w')) = x_2(H(w_1, w'))x_2(H(w_2, w')) ,$$

which is, in turn, equivalent to the fact that H is additive in the first variable. Completely similarly, ($\mathcal{A}_{24}$), ($\mathcal{B}_{13}$) and ($\mathcal{B}_{24}$) also follow from the fact that F and H are additive in both variables.

By ($\mathbf{Q}_{12}$), the left hand side of ($\mathcal{A}_{14}$) can be rewritten as

$$x_2(vw_1 + vw_2 + H(w_2, w_1v))x_3((w_1 \boxplus w_2)v) .$$

Using the fact that $b^a = [a, b^{-1}]b$, we can rewrite the right hand side as

$$x_2(vw_1)[x_1(w_2), x_3(w_1v)^{-1}]x_3(w_1v)x_2(vw_2)x_3(w_2v)$$

which is also equal to

$$x_2(vw_1)x_2(H(w_2, w_1v))x_3(w_1v)x_2(vw_2)x_3(w_2v) .$$

Since $[U_2, U_2] = [U_2, U_3] = 1$, we can rewrite this once more as

$$x_2(vw_1 + vw_2 + H(w_2, w_1v))x_3(w_1v + w_2v) .$$

It now follows from ($\mathbf{Q}_3$) that ($\mathcal{A}_{14}$) holds.

Similarly, ($\mathcal{B}_{14}$) follows from ($\mathbf{Q}_{11}$) and ($\mathbf{Q}_4$); we additionally need the fact that $\mathrm{Im}(F) \leq Z(W)$, which follows from ($\mathbf{Q}_7$) and ($\mathbf{Q}_8$).

Since $[U_2, U_1] = [U_2, U_2] = [U_2, U_3] = 1$, ($\mathcal{C}_{13}$) becomes trivial. Because $[U_3, U_2] = [U_3, U_4] = 1$, we have that ($\mathcal{C}_{24}$) is equivalent to the condition $[w, F(v, v')]_\boxplus = 1$. Since $\mathrm{Im}(F) \leq Z(W)$, this is always satisfied.

To prove ($\mathcal{C}_{14,2}$), we need to show that

$$x_2(v') = x_2(v')^{x_4(v)x_1(w)x_4(-v)} ,$$

which is the same as

$$(x_2(v')^{x_4(v)})^{x_1(w)} = x_2(v')^{x_4(v)} .$$

Since $x_2(v')^{x_4(v)} = x_2(v')[x_2(v'), x_4(-v)^{-1}] = x_2(v')x_3(F(v', -v))$, we have that

$$\begin{aligned}(x_2(v')^{x_4(v)})^{x_1(w)} &= (x_2(v')x_3(F(v', -v)))^{x_1(w)} \\ &= x_2(v')[x_1(w), x_3(F(v', -v))^{-1}]x_3(F(v', -v)) \\ &= x_2(v')x_2(H(w, F(v', -v)))x_3(F(v', -v)) \\ &= x_2(v')x_3(F(v', -v)) \\ &= x_2(v')^{x_4(v)}\end{aligned}$$

since $\mathrm{Im}(F) \leq \mathrm{Rad}(H)$ by $(\mathbf{Q}_7)$. Thus $(\mathcal{C}_{14,2})$ holds.

The left hand side of $(\mathcal{C}_{14,3})$ is equal to

$$x_3(w')[x_3(w'), x_3(wv)] ,$$

which is, by $(\mathbf{Q}_8)$, also equal to

$$x_3(w' \boxplus F(H(w, w'), v)) .$$

The right hand side is equal to

$$\begin{aligned}&x_3(w')^{x_1(\boxminus w)x_4(v)x_1(w)x_4(-v)} \\ &\qquad = (x_2(-H(w, w')) \cdot x_3(w'))^{x_4(v)x_1(w)x_4(-v)} \\ &\qquad = (x_2(-H(w, w')) \cdot x_3(F(H(w, w'), v))x_3(w'))^{x_1(w)x_4(-v)} \\ &\qquad = (x_2(-H(w, w')) \cdot x_3(w' \boxplus F(H(w, w'), v)))^{x_1(w)x_4(-v)} \\ &\qquad = (x_2(-H(w, w')) \cdot x_2(H(w, w' \boxplus F(H(w, w'), v))) \\ &\qquad\qquad \cdot x_3(w' \boxplus F(H(w, w'), v)))^{x_4(-v)} \\ &\qquad = x_3(w' \boxplus F(H(w, w'), v))^{x_4(-v)} \\ &\qquad = x_3(w' \boxplus F(H(w, w'), v)) ,\end{aligned}$$

thus $(\mathcal{C}_{14,3})$ holds. This concludes the proof of this theorem. □

It now follows Theorem 1.4.7 that the sequence $(U_+, U_1, U_2, U_3, U_4)$ satisfies the conditions $(\mathcal{M}_1)$ and $(\mathcal{M}_2)$.

Let U_0 be a group parametrized by V (via a map x_0), and let U_5 be a group parametrized by W (via a map x_5). We define an action of U_0 on

$U_{[1,3]}$ by the following commutator relations.

$$\begin{aligned}[][U_0, U_1] &= 1 \\ [x_0(v_1), x_2(v_2)^{-1}] &= x_1(F(v_1, \overline{v_2})) \\ [x_0(v), x_3(w)^{-1}] &= x_1(wv)x_2(-\overline{v(\boxminus w)})\end{aligned}$$

for all $w \in W$ and all $v, v_1, v_2 \in V$. For each $x_4(v) \in U_4^*$, we define an element $\mu(x_4(v)) \in U_0^* x_4(v) U_0^*$ as

$$\mu(x_4(v)) = x_0(v^{-1})x_4(v)x_0(v^{-1}) \ .$$

We define an action of U_5 on $U_{[2,4]}$ by the following commutator relations.

$$\begin{aligned}[][x_2(v), x_5(w)^{-1}] &= x_3(w(-v))x_4(-v(\boxminus w)) \\ [x_3(w_1), x_5(w_2)^{-1}] &= x_4(H(w_2, w_1)) \\ [U_4, U_5] &= 1\end{aligned}$$

for all $w, w_1, w_2 \in W$ and all $v \in V$. For each $x_1(w) \in U_1^*$, we define an element $\mu(x_1(w)) \in U_5^* x_1(w) U_5^*$ as

$$\mu(x_1(w)) = x_5(\kappa(w))x_1(w)x_5(\lambda(w)) \ .$$

Note that, by Lemma 2.2.10, $\mu(x_4(v)^{-1}) = \mu(x_4(v))^{-1}$, and $\mu(x_1(w)^{-1}) = \mu(x_1(w))^{-1}$.

In order to obtain a Moufang quadrangle, the graph $\mathcal{Q}(V, W, \tau_V, \tau_W, \epsilon, \delta)$ has to satisfy the conditions $(\mathcal{M}_3)$ and $(\mathcal{M}_4)$ of Theorem 1.4.4. In Theorem 2.3.2, we will show that $(\mathcal{M}_3)$ holds; the validity of $(\mathcal{M}_4)$ will be shown in Theorem 2.3.3.

Theorem 2.3.2. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then the corresponding graph $\mathcal{Q}(V, W, \tau_V, \tau_W, \epsilon, \delta)$, together with the group U_0 and the map μ, satisfies the following conditions, for all $v \in V$.*

(i) $U_0^{\mu(x_4(\epsilon))} = U_4$, *considered as subgroups of* $\mathrm{Aut}(U_{[1,3]})$*;*

(ii) $U_1^{\mu(x_4(v))} = U_3$. *More precisely, we have that* $x_1(w)^{\mu(x_4(v))} = x_3(w(-v))$ *for all $w \in W$ and all $v \in V^*$;*

(iii) $U_2^{\mu(x_4(v))} = U_2$. *More precisely, we have that* $x_2(v')^{\mu(x_4(v))} = x_2(\pi_v(v'))$ *for all $v' \in V$ and all $v \in V^*$;*

(iv) $U_3^{\mu(x_4(v))} = U_1$. *More precisely, we have that* $x_3(w)^{\mu(x_4(v))} = x_1(wv^{-1})$ *for all $w \in W$ and all $v \in V^*$;*

(v) $U_4^{\mu(x_4(\epsilon))} = U_0$, *considered as subgroups of* $\mathrm{Aut}(U_{[1,3]})$.

Proof. For all $w \in W$ and all $v \in V^*$, we have

$$\begin{aligned}
x_1(w)^{\mu(x_4(v))} &= x_1(w)^{x_0(v^{-1})x_4(v)x_0(v^{-1})} \\
&= x_1(w)^{x_4(v)x_0(v^{-1})} \\
&= (x_1(w)x_2(-vw)x_3(w(-v)))^{x_0(v^{-1})} \\
&= x_1(w)x_1(F(-\overline{vw}, v^{-1}))x_2(-vw)x_1(w(-v)v^{-1}) \\
&\qquad \cdot x_2(-\overline{v^{-1}(\boxminus w(-v))})x_3(w(-v)) \\
&= x_3(w(-v)) \ ,
\end{aligned}$$

where we have used Lemma 2.2.16, ($\mathbf{Q}_{16}$) and Lemma 2.2.12 for the last equality. By substituting wv^{-1} for w and $-v$ for v, we also get

$$x_1(wv^{-1})^{\mu(x_4(-v))} = x_3(wv^{-1}v) \ ,$$

and since $\mu(x_4(-v)) = \mu(x_4(v))^{-1}$ and by ($\mathbf{Q}_{15}$), it follows that

$$x_3(w)^{\mu(x_4(v))} = x_1(wv^{-1}) \ .$$

So we have proved that $U_1^{\mu(x_4(v))} \subseteq U_3$ and $U_3^{\mu(x_4(v))} \subseteq U_1$. If we replace v by $-v$ in those two relations, and conjugate by $\mu(x_4(v))$, it also follows that $U_1 \subseteq U_3^{\mu(x_4(v))}$ and $U_3 \subseteq U_1^{\mu(x_4(v))}$. So (ii) and (iv) are proved.

We will now prove (iii). For all $v \in V^*$ and all $v' \in V$, we have

$$\begin{aligned}
x_2(v')^{\mu(x_4(v))} &= x_2(v')^{x_0(v^{-1})x_4(v)x_0(v^{-1})} \\
&= (x_1(F(v^{-1}, \overline{v'}))x_2(v'))^{x_4(v)x_0(v^{-1})} \\
&= (x_1(F(v^{-1}, \overline{v'}))x_2(-vF(v^{-1}, \overline{v'}))x_3(F(v^{-1}, \overline{v'})(-v)) \\
&\qquad \cdot x_2(v')x_3(\boxminus F(v', v)))^{x_0(v^{-1})} \\
&= (x_1(F(v^{-1}, \overline{v'}))x_2(v' - vF(v^{-1}, \overline{v'})))^{x_0(v^{-1})} \ ,
\end{aligned}$$

where we have used ($\mathbf{Q}_{17}$) for the last equality. It follows that

$$\begin{aligned}
x_2(v')^{\mu(x_4(v))} &= x_1(F(v^{-1}, \overline{v'}))x_1(F(v^{-1}, \overline{v' - vF(v^{-1}, \overline{v'})})) \\
&\qquad \cdot x_2(v' - vF(v^{-1}, \overline{v'})) \\
&= x_2(v' - vF(v^{-1}, \overline{v'})) \\
&= x_2(\pi_v(v')) \ ,
\end{aligned}$$

where we have used Lemma 2.2.9 and Theorem 2.2.17(i). It follows that $U_2^{\mu(x_4(v))} \subseteq U_2$, and again by replacing v by $-v$ and conjugating by $\mu(x_4(v))$, we get that $U_2 \subseteq U_2^{\mu(x_4(v))}$ as well, from which (iii) follows.

To prove (v), we will check that the action of $\mu(x_4(-\epsilon))x_4(v)\mu(x_4(\epsilon))$ on $U_{[1,3]}$ is the same as the action of $x_0(v)$ on $U_{[1,3]}$, for all $v \in V$. Note that we will use the fact that $w\epsilon^{-1} = w$, which follows by choosing $v = \epsilon$ in ($\mathbf{Q}_{15}$), and the fact that $F(\epsilon^{-1}, \overline{v}) = F(\epsilon, v)$, which holds by substituting ϵ for v_1 in ($\mathbf{Q}_{17}$).

Using the definition of the map $v \mapsto \overline{v}$, we see that

$$\begin{aligned} x_2(v)^{\mu(x_4(\epsilon))} &= x_2(v - \epsilon F(\epsilon^{-1}, \overline{v})) \\ &= x_2(v - \epsilon F(\epsilon, v)) \\ &= x_2(-\overline{v}) \ , \end{aligned}$$

for all $v \in V$. Since $-\overline{(-\overline{v})} = v$, replacing v by $-\overline{v}$ and conjugating by $\mu(x_4(-\epsilon))$ yields

$$x_2(v)^{\mu(x_4(-\epsilon))} = x_2(-\overline{v})$$

for all $v \in V$, as well. For the action on U_1, we have

$$\begin{aligned} x_1(w)^{\mu(x_4(-\epsilon))x_4(v)\mu(x_4(\epsilon))} &= x_3(w)^{x_4(v)\mu(x_4(\epsilon))} \\ &= x_3(w)^{\mu(x_4(\epsilon))} \\ &= x_1(w) \\ &= x_1(w)^{x_0(v)} \ ; \end{aligned}$$

for the action on U_2, we have

$$\begin{aligned} x_2(v')^{\mu(x_4(-\epsilon))x_4(v)\mu(x_4(\epsilon))} &= x_2(-\overline{v'})^{x_4(v)\mu(x_4(\epsilon))} \\ &= (x_2(-\overline{v'})x_3(F(\overline{v'}, v)))^{\mu(x_4(\epsilon))} \\ &= x_2(v')x_1(F(v, \overline{v'})) \\ &= x_2(v')^{x_0(v)} \ . \end{aligned}$$

To check the action on U_3, we need ($\mathbf{Q}_6$), ($\mathbf{Q}_5$), ($\mathbf{Q}_{22}$), Lemma 2.2.9 and Lemma 2.2.13(ii) :

$$\begin{aligned} x_3(w)^{\mu(x_4(-\epsilon))x_4(v)\mu(x_4(\epsilon))} &= x_1(w(-\epsilon))^{x_4(v)\mu(x_4(\epsilon))} \\ &= (x_1(w(-\epsilon))x_2(-vw)x_3(wv))^{\mu(x_4(\epsilon))} \\ &= x_3(w)x_2(\overline{vw})x_1(wv) \\ &= x_1(wv)x_2(\overline{vw} + H(wv, w))x_3(w) \\ &= x_1(wv)x_2(\overline{vw - H(w, wv)})x_3(w) \\ &= x_1(wv)x_2(-\overline{v(\boxminus w)})x_3(w) \\ &= x_3(w)^{x_0(v)} \ . \end{aligned}$$

Thus (v) is proved.

To prove (i), we will check that the action of $\mu(x_4(\epsilon))x_4(v)\mu(x_4(-\epsilon))$ on $U_{[1,3]}$ is the same as the action of $x_0(v)$ on $U_{[1,3]}$. We can take a shortcut by observing that

$$\mu(x_4(\epsilon))x_4(v)\mu(x_4(-\epsilon)) = \mu(x_4(\epsilon))^2\mu(x_4(-\epsilon))x_4(v)\mu(x_4(\epsilon))\mu(x_4(-\epsilon))^2 .$$

We just have to do a short calculation to see that

$$\begin{aligned} x_1(w)^{\mu(x_4(\epsilon))^2} &= x_1(w)^{\mu(x_4(-\epsilon))^2} = x_1(w(-\epsilon)) ; \\ x_2(v)^{\mu(x_4(\epsilon))^2} &= x_2(v)^{\mu(x_4(-\epsilon))^2} = x_2(v) ; \\ x_3(w)^{\mu(x_4(\epsilon))^2} &= x_3(w)^{\mu(x_4(-\epsilon))^2} = x_3(w(-\epsilon)) . \end{aligned}$$

For the action on U_1, we have

$$\begin{aligned} x_1(w)^{\mu(x_4(\epsilon))x_4(v)\mu(x_4(-\epsilon))} &= x_1(w(-\epsilon))^{\mu(x_4(-\epsilon))x_4(v)\mu(x_4(\epsilon))\mu(x_4(-\epsilon))^2} \\ &= x_1(w(-\epsilon))^{\mu(x_4(-\epsilon))^2} \\ &= x_1(w) \\ &= x_1(w)^{x_0(v)} ; \end{aligned}$$

for the action on U_2, we have, by Lemma 2.2.15, that

$$\begin{aligned} x_2(v')^{\mu(x_4(\epsilon))x_4(v)\mu(x_4(-\epsilon))} &= x_2(v')^{\mu(x_4(-\epsilon))x_4(v)\mu(x_4(\epsilon))\mu(x_4(-\epsilon))^2} \\ &= (x_2(v')x_1(F(v,\overline{v'})))^{\mu(x_4(-\epsilon))^2} \\ &= x_2(v')x_1(F(v,\overline{v'})(-\epsilon)) \\ &= x_2(v')x_1(F(v,\overline{v'})) \\ &= x_2(v')^{x_0(v)} ; \end{aligned}$$

Finally, for the action on U_3, we have, using Lemma 2.2.11, that

$$\begin{aligned} x_3(w)^{\mu(x_4(\epsilon))x_4(v)\mu(x_4(-\epsilon))} &= x_3(w(-\epsilon))^{\mu(x_4(-\epsilon))x_4(v)\mu(x_4(\epsilon))\mu(x_4(-\epsilon))^2} \\ &= (x_1(w(-v))x_2(-\overline{v(\boxminus w)})x_3(w(-\epsilon)))^{\mu(x_4(-\epsilon))^2} \\ &= x_1(wv)x_2(-\overline{v(\boxminus w)})x_3(w) \\ &= x_3(w)^{x_0(v)} . \end{aligned}$$

So we have proved (i), and this completes the proof of this theorem. □

Theorem 2.3.3. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then the corresponding graph $\mathcal{Q}(V, W, \tau_V, \tau_W, \epsilon, \delta)$, together with the group U_5 and the map μ, satisfies the following conditions, for all $w \in W$.*

(i) $U_5^{\mu(x_1(\delta))} = U_1$, *considered as subgroups of* $\mathrm{Aut}(U_{[2,4]})$;

(ii) $U_4^{\mu(x_1(w))} = U_2$. *More precisely, we have that* $x_4(v)^{\mu(x_1(w))} = x_2(vw)$ *for all* $v \in V$ *and all* $w \in W^*$;

(iii) $U_3^{\mu(x_1(w))} = U_3$. *More precisely, we have that* $x_3(w')^{\mu(x_1(w))} = x_3(\Pi_w(w'))$ *for all* $w' \in W$ *and all* $w \in W^*$;

(iv) $U_2^{\mu(x_1(w))} = U_4$. *More precisely, we have that* $x_2(v)^{\mu(x_1(w))} = x_4(-v\kappa(w))$ *for all* $v \in V$ *and all* $w \in W^*$;

(v) $U_1^{\mu(x_1(\delta))} = U_5$, *considered as subgroups of* $\mathrm{Aut}(U_{[2,4]})$.

Proof. The proof of this theorem is very similar to the previous one, so we will skip most of the calculations.

For all $w \in W^*$ and all $v \in V$, we have

$$\begin{aligned} x_2(v)^{\mu(x_1(w))} &= x_2(v)^{x_5(\kappa(w))x_1(w)x_5(\lambda(w))} \\ &= x_4(-v\kappa(w)) \ , \end{aligned}$$

where we have to use ($\mathbf{Q}_{19}$) and Lemma 2.2.16(ii). By substituting $-w$ for w and vw for v, we also get

$$x_2(vw)^{\mu(x_1(\boxminus w))} = x_4(-vw\kappa(\boxminus w)) \ ,$$

and since $\mu(x_1(\boxminus w)) = \mu(x_1(w))^{-1}$ and by Lemma 2.2.2(i), it follows that

$$x_4(v)^{\mu(x_1(w))} = x_2(vw) \ .$$

So we have proved that $U_4^{\mu(x_1(w))} \subseteq U_2$ and $U_2^{\mu(x_1(w))} \subseteq U_4$. If we replace w by $\boxminus w$ in those two relations, and conjugate by $\mu(x_1(w))$, it also follows that $U_4 \subseteq U_2^{\mu(x_1(w))}$ and $U_2 \subseteq U_4^{\mu(x_1(w))}$. So (ii) and (iv) are proved.

We will now prove (iii). For all $w \in W^*$ and all $w' \in W$, we have

$$\begin{aligned} x_3(w')^{\mu(x_1(w))} &= x_3(w')^{x_5(\kappa(w))x_1(w)x_5(\lambda(w))} \\ &= x_3(w' \boxplus w(-H(\kappa(w), w'))) \\ &= x_3(\Pi_w(w')) \ , \end{aligned}$$

where we have to use ($\mathbf{Q}_{20}$) and Theorem 2.2.17(ii). Hence $U_3^{\mu(x_1(w))} \subseteq U_3$, and again by replacing w by $-w$ and conjugating by $\mu(x_1(w))$, we get that $U_3 \subseteq U_3^{\mu(x_1(w))}$ as well, from which (iii) follows.

To prove (v), we will check that the action of $\mu(x_1(\boxminus\delta))x_1(w)\mu(x_1(\delta))$ on $U_{[2,4]}$ is the same as the action of $x_5(w)$ on $U_{[2,4]}$, for all $w \in W$. First of all, observe that it follows from Lemma 2.2.13(ii) and from the fact that

$\delta \in \mathrm{Rad}(H)$ (by (**Q**$_7$)) that $v(\boxminus\delta) = -v$. If we put $w = \delta$ in (**Q**$_{18}$), it thus follows that $v\kappa(\delta) = v$; if we put $w = \boxminus\delta$ in this same identity (**Q**$_{18}$), it follows that $v\kappa(\boxminus\delta) = -v$. Furthermore, if we put $w_1 = \delta$ in (**Q**$_{20}$), it follows from (**Q**$_7$) that $H(\kappa(\delta), w) = 0$, for all $w \in W$.

Using these facts, we can prove that

$$\begin{aligned} x_4(v)^{\mu(x_1(\boxminus\delta))x_1(w)\mu(x_1(\delta))} &= x_4(v)^{x_5(w)} ; \\ x_3(w')^{\mu(x_1(\boxminus\delta))x_1(w)\mu(x_1(\delta))} &= x_3(w')^{x_5(w)} ; \\ x_2(v)^{\mu(x_1(\boxminus\delta))x_1(w)\mu(x_1(\delta))} &= x_2(v)^{x_5(w)} , \end{aligned}$$

where we have to use (**Q**$_{21}$) and Lemma 2.2.13(i) as well. Thus (v) is proved.

To prove (i), we have to check that the action of $\mu(x_1(\delta))x_5(w)\mu(x_1(\boxminus\delta))$ on $U_{[2,4]}$ is the same as the action of $x_5(w(-\epsilon))$ on $U_{[2,4]}$. Again, we can take a shortcut by observing that

$$\mu(x_1(\delta))x_1(w)\mu(x_1(\boxminus\delta)) = \mu(x_1(\delta))^2\mu(x_1(\boxminus\delta))x_1(w)\mu(x_1(\delta))\mu(x_1(\boxminus\delta))^2 .$$

First, we observe that

$$\begin{aligned} x_2(v)^{\mu(x_1(\delta))^2} &= x_2(v)^{\mu(x_1(\boxminus\delta))^2} = x_2(-v) ; \\ x_3(w)^{\mu(x_1(\delta))^2} &= x_3(w)^{\mu(x_1(\boxminus\delta))^2} = x_3(w) ; \\ x_4(v)^{\mu(x_1(\delta))^2} &= x_4(v)^{\mu(x_1(\boxminus\delta))^2} = x_4(-v) . \end{aligned}$$

It now follows from a short calculation that

$$\begin{aligned} x_4(v)^{\mu(x_1(\delta))x_5(w)\mu(x_1(\boxminus\delta))} &= x_4(v)^{x_5(w(-\epsilon))} ; \\ x_3(w')^{\mu(x_1(\delta))x_5(w)\mu(x_1(\boxminus\delta))} &= x_3(w')^{x_5(w(-\epsilon))} ; \\ x_2(v)^{\mu(x_1(\delta))x_5(w)\mu(x_1(\boxminus\delta))} &= x_2(v)^{x_5(w(-\epsilon))} . \end{aligned}$$

So we have proved (i), and this completes the proof of this theorem. □

This completes the proof of the fact that the graph $\mathcal{Q}(V, W, \tau_V, \tau_W, \epsilon, \delta)$ is a Moufang quadrangle.

2.4 From Moufang Quadrangles To Quadrangular Systems

In this section, we will prove that *every* Moufang quadrangle can be obtained from the construction described in the previous sections. We will make intensive use of Chapter 21 "Quadrangles" in [52]. Since we are dealing with the same objects as in [52], it should not be very surprising that we need these same properties. However, after recalling these facts, our approach will very quickly diverge from the one given in [52].

Let Γ be an arbitrary Moufang quadrangle. As in section 1.4, we will fix an apartment $\Sigma = (0, 1, \dots, 7)$, where the vertices are labeled modulo 8, and we will denote its root groups $U(i, i+1, \dots, i+4)$ by U_i, for all $i \in \mathbb{Z}$.

Let $V_i := [U_{i-1}, U_{i+1}] \leq U_i$ and $Y_i := C_{U_i}(U_{i-2}) \leq U_i$ for each i. It can be shown (see [52, (21.20.i)]) that $Y_i = C_{U_i}(U_{i+2})$ as well.

The following theorem defines the functions κ, λ and μ.

Theorem 2.4.1. *For each i, there exist unique functions $\kappa_i, \lambda_i : U_i^* \to U_{i+4}^*$, such that $(i-1)^{a_i \lambda_i(a_i)} = i+1$ and $(i+1)^{\kappa_i(a_i) a_i} = i-1$, for all $a_i \in U_i^*$. The product $\mu_i(a_i) := \kappa_i(a_i) a_i \lambda_i(a_i)$ fixes i and $i+4$ and reflects Σ, and $U_j^{\mu_i(a_i)} = U_{2i+4-j}$ for each $a_i \in U_i^*$ and each j.*

Proof. See [52, (6.1)]. □

Since we will apply these functions only when it is clear in which U_i^* the argument lies, we will write κ, λ and μ in place of κ_i, λ_i and μ_i. Note that it follows from the last statement of this theorem, that U_i and U_j are conjugate (and hence isomorphic) whenever i and j have the same parity.

Lemma 2.4.2. *For all $a_i \in U_i^*$, we have :*

(i) $\mu(a_i^{-1}) = \mu(a_i)^{-1}$;

(ii) $\lambda(a_i^{-1}) = \kappa(a_i)^{-1}$;

(iii) $\mu(a_i^g) = \mu(a_i)^g$ *for every element $g \in \mathrm{Aut}(\Gamma)$ mapping Σ to itself.*

Proof. See [52, (6.2)]. □

The following "Shift Lemma" is essential.

Theorem 2.4.3. *Suppose, for some i, that $[a_i, a_{i+3}^{-1}] = a_{i+1} a_{i+2}$, with $a_k \in U_k$ for each k, and with a_i and a_{i+3} non-trivial. Then we have:*

(i) $a_i = a_{i+2}^{\mu(a_{i+3})}$ *and* $a_{i+1} = a_{i+3}^{\mu(a_i)}$;

(ii) $[\kappa(a_{i+3}), a_{i+2}^{-1}] = a_i a_{i+1}$;

(iii) $[a_{i+1}, \lambda(a_i)^{-1}] = a_{i+2} a_{i+3}$.

Proof. See [52, (21.19)]. □

The following theorem already puts strong restrictions on the root groups.

Theorem 2.4.4. *By relabeling the vertices of Σ by the transformation $i \mapsto 5-i$ if necessary, we can assume the following :*

(i) $Y_i \neq 1$, $[U_i, U_i] \leq V_i \leq Y_i \leq Z(U_i)$, *for all odd i ;*

(ii) U_i *is abelian, for all even i .*

Proof. See [52, (21.28)]. □

From now on, we will assume that we have chosen the labeling of our apartment Σ in such a way that the statements of Theorem 2.4.4 hold.
We will also use the following results from [52].

Theorem 2.4.5. *(see [52, (21.29)])*
If $a_1 \in Y_1^*$, then $\kappa(a_1)$ and $\lambda(a_1)$ both lie in Y_5^*.

Theorem 2.4.6. *(see [52, (21.33)])*
Let $h = \mu(a_1)^2$, for some $a_1 \in Y_1^*$. Then :

(i) $a_3^h = a_3$, *for all $a_3 \in U_3$;*

(ii) $a_4^h = a_4^{-1}$, *for all $a_4 \in U_4$.*

Theorem 2.4.7. *(see [52, (21.34)])*
$\kappa(a_4) = \lambda(a_4)$, *for all $a_4 \in U_4^*$.*

Theorem 2.4.8. *(see [52, (21.36)])*
Let $a_1 \in U_1^*$, $a_2 \in U_2$, $a_3 \in U_3$ and $a_4 \in U_4^*$. Then :

(i) $a_2^{\mu(a_4)} a_2^{-1} = [[\lambda(a_4), a_2^{-1}], a_4]_2$;

(ii) $[[\lambda(a_4), a_2^{-1}], a_4]_3 = [a_2, a_4]^{-1}$;

(iii) $[a_1, [a_3, \kappa(a_1)]^{-1}]_2 = [a_1, a_3^{-1}]^{-1}$.

Theorem 2.4.9. *(see [52, (21.37)])*
$[\mu(a_4)^2, Y_1 U_2 Y_3 U_4] = 1$, *for all $a_4 \in U_4^*$.*

Proof. For all the proofs of these theorems, see [52], except for Theorem 2.4.8(iii), for which the proof is completely similar to the proof of Theorem 2.4.8(ii). □

We can now start to build up our quadrangular systems. We start the construction by choosing an arbitrary parametrization of the group U_1 by some group $(W, \boxplus) \cong U_1$, and an arbitrary parametrization of the group U_4 by some group $(V, +) \cong U_4$. We will denote the isomorphisms from W to U_1 and from V to U_4 by x_1 and x_4, respectively. Choose some fixed elements $e_1 = x_1(\delta) \in Y_1^*$ (note that Y_1^* is non-empty because of Theorem 2.4.4(i)) and $e_4 = x_4(\epsilon) \in U_4^*$, where we choose e_4 in Y_4^* if $Y_4 \neq 1$. Since

U_3 is isomorphic to U_1, we can also have it parametrized by the same group $(W, \boxplus)$ by some isomorphism x_3, which we define by setting

$$x_3(w) := [x_1(w), e_4^{-1}]_3 \,,$$

for all $w \in W$. Similarly, we let U_2 be parametrized by $(V, +)$, via the isomorphism x_2 defined by

$$x_2(v) := [e_1, x_4(v)^{-1}]_2 \,,$$

for all $v \in V$. To parametrize U_0 and U_5, we choose the following isomorphisms x_0 and x_5 from V to U_0 and from W to U_5, respectively :

$$x_0(v) := x_4(v)^{\mu(e_4)} \,,$$
$$x_5(w) := x_1(w)^{\mu(e_1)} \,,$$

for all $w \in W$ and all $v \in V$. We will now define a map F from $V \times V$ to W and a map H from $W \times W$ on V, by setting

$$[x_1(w_1), x_3(w_2)^{-1}] = x_2(H(w_1, w_2)) \,,$$
$$[x_2(v_1), x_4(v_2)^{-1}] = x_3(F(v_1, v_2)) \,,$$

for all $w_1, w_2 \in W$ and all $v_1, v_2 \in V$. Furthermore, we define a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W, both of which will be denoted by $\cdot$ or by juxtaposition, by setting

$$[x_1(w), x_4(v)^{-1}]_2 = x_2(\tau_V(v, w)) = x_2(vw) \,,$$
$$[x_1(w), x_4(v)^{-1}]_3 = x_3(\tau_W(w, v)) = x_3(wv) \,,$$

for all $w \in W$ and all $v \in V$. Finally, for each $w \in W^*$, we define two elements $\kappa(w), \lambda(w) \in W^*$ by setting

$$\kappa(x_1(w)) = x_5(\kappa(w)) \,,$$
$$\lambda(x_1(w)) = x_5(\lambda(w)) \,,$$

and for each $v \in V^*$, we define an element $v^{-1} \in V^*$, by setting

$$\kappa(x_4(v)) = x_0(v^{-1}) \,.$$

Note that, by Theorem 2.4.7, $\lambda(x_4(v)) = x_0(v^{-1})$ as well.

If we can now prove that these data satisfy all of the axioms $(\mathbf{Q}_1)$ – $(\mathbf{Q}_{20})$, then we have proved that every Moufang quadrangle can actually be obtained from the construction in the previous sections, since we have started from an arbitrary Moufang quadrangle. At the same time, however, we will show that the identities $(\mathbf{Q}_{21})$ – $(\mathbf{Q}_{26})$ hold; see Theorem 2.5.1.

Remark 2.4.10. It is interesting to observe that the choice of δ and ϵ is arbitrary (up to some restrictions about the radical). This gives us some freedom for the choice of the base points for the parametrizing structure of an arbitrary Moufang quadrangle. See also Remark 2.5.4.

By Theorem 2.4.4(ii), the group U_4 is abelian. Since U_4 is parametrized by $(V,+)$, we have that V is abelian as well.

By the definition of the isomorphism x_3 and the definition of the map from $V \times W$ to V, we have, for all $w \in W$, that $x_3(w) = [x_1(w), e_4^{-1}]_3 = [x_1(w), x_4(\epsilon)^{-1}]_3 = x_3(w\epsilon)$, from which it follows that $w = w\epsilon$, which proves (**Q**$_1$). Similarly, we can prove that (**Q**$_2$) holds.

We now take a look at the subgroups V_3 and Y_3 of U_3. By definition, we have $V_3 = [U_2, U_4] = [x_2(V), x_4(V)^{-1}] = x_3(F(V,V)) = x_3(\mathrm{Im}(F))$. The elements of U_3 which commute with every element of U_1, are exactly those elements $x_3(w) \in U_3$ such that $[x_1(w'), x_3(w)] = 1$, for all $w' \in W$, this is, such that $x_2(H(w', w)) = 1$ or equivalently $H(w', w) = 0$, for all $w' \in W$. This means that $Y_3 = C_{U_3}(U_1) = x_3(\mathrm{Rad}(H))$. It now follows from Theorem 2.4.4(i) that $[W, W] \leq \mathrm{Im}(F) \leq \mathrm{Rad}(H) \leq Z(W)$. In particular, we have proved (**Q**$_7$). We have also proved that $[\mathrm{Im}(F), W] = 1$.

Completely similarly as in the previous paragraph, it follows from the definitions that $Y_1 = C_{U_1}(U_3) = x_1(\mathrm{Rad}(H))$ and that $Y_4 = C_{U_4}(U_2) = x_4(\mathrm{Rad}(F))$. It thus follows from $e_1 = x_1(\delta) \in Y_1^*$ that $\delta \in \mathrm{Rad}(H)^*$, and it follows from the fact that $e_4 = x_4(\epsilon)$ was chosen to lie in Y_4^* if $Y_4 \neq 1$ that $\epsilon \in \mathrm{Rad}(F)^*$ if $\mathrm{Rad}(F) \neq 0$. Hence we have shown (**Q**$_9$) and (**Q**$_{10}$).

Using the identity $[ab, c^{-1}] = [a, c^{-1}]^b [b, c^{-1}]$ and the fact that U_1 and U_2 commute (because of Theorem 1.4.1(i)), we can deduce that

$$\begin{aligned} x_2(H(w_1 \boxplus w_2, w')) &= [x_1(w_1 \boxplus w_2), x_3(w')^{-1}] \\ &= [x_1(w_1)x_1(w_2), x_3(w')^{-1}] \\ &= [x_1(w_1), x_3(w')^{-1}]^{x_1(w_2)} [x_1(w_2), x_3(w')^{-1}] \\ &= x_2(H(w_1, w'))^{x_1(w_2)} x_2(H(w_2, w')) \\ &= x_2(H(w_1, w')) x_2(H(w_2, w')) \\ &= x_2(H(w_1, w') + H(w_2, w')) \ , \end{aligned}$$

for all $w_1, w_2, w' \in W$, so H is additive in the first variable. Similarly, it follows from the identity $[a, (bc)^{-1}] = [a, b^{-1}][a, c^{-1}]^{b^{-1}}$ that H is additive in the second variable. In the same way, we can deduce from those two identities that F is additive in both variables. Since we will use this fact very often from now on, we will not mention it explicitly anymore.

Using the same identity $[ab, c^{-1}] = [a, c^{-1}]^b [b, c^{-1}]$ and the fact that $[U_2, U_2] = 1$ (since V is abelian) and $[U_2, U_3] = 1$ (by Theorem 1.4.1(i)), we deduce that

$$\begin{aligned}
x_2(v(w_1 \boxplus w_2))x_3((w_1 \boxplus w_2)v) \\
&= [x_1(w_1 \boxplus w_2), x_4(v)^{-1}] \\
&= [x_1(w_1)x_1(w_2), x_4(v)^{-1}] \\
&= [x_1(w_1), x_4(v)^{-1}]^{x_1(w_2)}[x_1(w_2), x_4(v)^{-1}] \\
&= (x_2(vw_1)x_3(w_1v))^{x_1(w_2)}x_2(vw_2)x_3(w_2v) \\
&= x_2(vw_1)x_2(H(w_2, w_1v))x_3(w_1v)x_2(vw_2)x_3(w_2v) \\
&= x_2(vw_1 + vw_2 + H(w_2, w_1v))x_3(w_1v \boxplus w_2v) ,
\end{aligned}$$

for all $w_1, w_2 \in W$ and all $v_1, v_2 \in V$. By Theorem 1.4.1(ii), this implies that

$$\begin{aligned}
&x_2(v(w_1 \boxplus w_2)) = x_2(vw_1 + vw_2 + H(w_2, w_1v)) \quad \text{and} \\
&x_3((w_1 \boxplus w_2)v) = x_3(w_1v \boxplus w_2v) ,
\end{aligned}$$

for all $w_1, w_2 \in W$ and all $v_1, v_2 \in V$, from which it follows that ($\mathbf{Q}_{12}$) and ($\mathbf{Q}_3$) hold.

Similarly, it follows from the identity $[a,(bc)^{-1}] = [a, b^{-1}][a, c^{-1}]^{b^{-1}}$, the fact that $[\mathrm{Im}(F), W] = 1$ and the fact that $[U_2, U_3] = 1$ (because of Theorem 1.4.1(i)), that ($\mathbf{Q}_{11}$) and ($\mathbf{Q}_4$) hold.

Now, we will *define* a map $v \mapsto \overline{v}$ from V to V, by setting

$$x_2(v)^{\mu(e_4)} = x_2(-\overline{v}) ,$$

for all $v \in V$; we will prove later on (see page 57) that $\overline{v} = \epsilon F(\epsilon, v) - v$. Note that, by Theorem 2.4.9, we have that $x_2(v)^{\mu(e_4)^2} = x_2(v)$, and hence $-\overline{(-\overline{v})} = v$, for all $v \in V$. If we invert the identity $x_2(v)^{\mu(e_4)} = x_2(-\overline{v})$, then we get $x_2(-v)^{\mu(e_4)} = x_2(\overline{v})$; it follows that $\overline{-v} = -\overline{v}$, for all $v \in V$. Combining these two relations, we also get $\overline{\overline{v}} = v$, for all $v \in V$.

Theorem 2.4.11. *For all $w \in W$ and all $v \in V$, we have:*

(i) $x_0(v)^{\mu(e_4)} = x_4(v)$;

(ii) $x_1(w)^{\mu(e_4)} = x_3(w(-\epsilon))$;

(iii) $x_2(v)^{\mu(e_4)} = x_2(-\overline{v})$;

(iv) $x_3(w)^{\mu(e_4)} = x_1(w)$;

(v) $x_4(v)^{\mu(e_4)} = x_0(v)$;

(vi) $x_1(w)^{\mu(e_1)} = x_5(w)$;

(vii) $x_2(v)^{\mu(e_1)} = x_4(-v)$;

(viii) $x_3(w)^{\mu(e_1)} = x_3(w)$;

(ix) $x_4(v)^{\mu(e_1)} = x_2(v)$;

(x) $x_5(w)^{\mu(e_1)} = x_1(w(-\epsilon))$.

Proof. First of all, (iii), (v) and (vi) hold by definition. By Theorem 2.4.9, $x_4(v)^{\mu(e_4)^2} = x_4(v)$. So if we conjugate (v) by $\mu(e_4)$, we get (i). If we apply Theorem 2.4.3(i) on the identity

$$[x_1(w), x_4(v)^{-1}] = x_2(vw)x_3(wv) ,$$

we get that

$$\begin{aligned} x_1(w) &= x_3(wv)^{\mu(x_4(v))} \quad \text{and} \\ x_2(vw) &= x_4(v)^{\mu(x_1(w))} , \end{aligned}$$

for all $w \in W^*$ and all $v \in V^*$. If we choose $v = \epsilon$ in the first equality, we get, by ($\mathbf{Q}_1$), that $x_1(w) = x_3(w)^{\mu(e_4)}$, which proves (iv). If we choose $v = -\epsilon$ in this same equality, we get

$$\begin{aligned} x_1(w) &= x_3(w(-\epsilon))^{\mu(e_4^{-1})} \\ &= x_3(w(-\epsilon))^{\mu(e_4)^{-1}} , \end{aligned}$$

by Lemma 2.4.2(i); conjugating by $\mu(e_4)$ yields (ii).

If we choose $w = \delta$ in the second equality, then it follows from ($\mathbf{Q}_2$) that $x_2(v) = x_4(v)^{\mu(e_1)}$, which proves (ix). By Theorem 2.4.6(ii), we have that $x_4(v)^{\mu(e_1)^2} = x_4(-v)$. So if we conjugate (ix) by $\mu(e_1)$, we get (vii).

By Theorem 2.4.5, we know that $\mu(e_1) \in Y_5Y_1Y_5$. Since $Y_1 = C_{U_1}(U_3)$ and $Y_5 = C_{U_5}(U_3)$, it follows that $[\mu(e_1), U_3] = 1$, which implies (viii).

If we conjugate the identity

$$[x_1(w), x_4(v)^{-1}] = x_2(vw)x_3(wv)$$

by $\mu(e_1)^2$, we get, using (vi), (vii), (viii) and (ix), that

$$[x_5(w)^{\mu(e_1)}, x_4(-v)^{-1}] = x_2(-vw)x_3(wv) ,$$

for all $w \in W$ and all $v \in V$. If we choose $v = -\epsilon$, then this yields

$$[x_5(w)^{\mu(e_1)}, e_4^{-1}] = x_2(\epsilon w)x_3(w(-\epsilon)) .$$

It now follows from Theorem 2.4.3(i) and (iv) that

$$x_5(w)^{\mu(e_1)} = x_3(w(-\epsilon))^{\mu(e_4)} = x_1(w(-\epsilon)) ,$$

for all $w \in W$, which proves (x). $\square$

So far, we have proved ($\mathbf{Q}_1$), ($\mathbf{Q}_2$), ($\mathbf{Q}_3$), ($\mathbf{Q}_4$), ($\mathbf{Q}_7$), ($\mathbf{Q}_9$), ($\mathbf{Q}_{10}$), ($\mathbf{Q}_{11}$) and ($\mathbf{Q}_{12}$). We now continue to prove the other axioms.

If we conjugate the identity

$$[x_1(w_1), x_3(w_2)^{-1}] = x_2(H(w_1, w_2))$$

by $\mu(e_4)$, we get, by Theorem 2.4.11, that

$$[x_3(w_1(-\epsilon)), x_1(w_2)^{-1}] = x_2(-\overline{H(w_1, w_2)}) ,$$

for all $w_1, w_2 \in W$. Using the fact that $[b, a] = [a, b]^{-1}$, it follows that

$$[x_1(\boxminus w_2), x_3(\boxminus w_1(-\epsilon))^{-1}] = x_2(\overline{H(w_1, w_2)}) ,$$

hence

$$x_2(H(\boxminus w_2, \boxminus w_1(-\epsilon))) = x_2(\overline{H(w_1, w_2)}) ,$$

for all $w_1, w_2 \in W$. Using the fact that H is additive, it follows from this last equality that $H(w_2, w_1(-\epsilon)) = \overline{H(w_1, w_2)}$, for all $w_1, w_2 \in W$. Note that it follows from ($\mathbf{Q}_{12}$) that $H(w_2, w_1(-\epsilon)) = -H(w_2, w_1)$, for all $w_1, w_2 \in W$, so we have that $-H(w_2, w_1) = \overline{H(w_1, w_2)}$, which proves ($\mathbf{Q}_{22}$).

Completely similarly, we can conjugate the identity

$$[x_2(v_1), x_4(v_2)^{-1}] = x_3(F(v_1, v_2))$$

by $\mu(e_1)$, and, again by Theorem 2.4.11, we find after a short calculation that $F(v_1, v_2) = F(v_2, v_1)$, for all $v_1, v_2 \in V$, which proves ($\mathbf{Q}_{21}$).

If we conjugate the identity

$$[x_1(w), x_4(-v)^{-1}] = x_2(-vw)x_3(w(-v))$$

by $\mu(e_1)^2$, then we get, by Theorem 2.4.11, that

$$[x_1(w(-\epsilon)), x_4(v)^{-1}] = x_2(vw)x_3(w(-v)) ,$$

for all $w \in W$ and all $v \in V$. But on the other hand, we have that

$$[x_1(w(-\epsilon)), x_4(v)^{-1}] = x_2(v(w(-\epsilon)))x_3(w(-\epsilon)v) ,$$

for all $w \in W$ and all $v \in V$. By Theorem 1.4.1(ii), this implies that $vw = v(w(-\epsilon))$ and $w(-v) = w(-\epsilon)v$, for all $w \in W$ and all $v \in V$. Thus we have proved ($\mathbf{Q}_6$) and ($\mathbf{Q}_5$).

If we conjugate the same identity

$$[x_1(w), x_4(v)^{-1}] = x_2(vw)x_3(wv)$$

by $\mu(e_4)^2$, we get, again by Theorem 2.4.11, that

$$[x_1(w(-\epsilon)), x_4(v)^{-1}] = x_2(vw)x_3(wv(-\epsilon)) ,$$

from which it follows immediately (by Theorem 1.4.1(ii)) that $w(-\epsilon)v = wv(-\epsilon)$, for all $w \in W$ and all $v \in V$. This means that $w(-v) = wv(-\epsilon)$ as well.

We will now prove ($\mathbf{Q}_8$). We will make use of the identity $[a,b] = a^{-1}a^b$ and of the identity $[abc,d] = [a,d]^{bc}[b,d]^c[c,d]$.

$$\begin{aligned}
&x_3(F(H(w_2,w_1),v)) \\
&\qquad = [x_2(H(w_2,w_1)), x_4(v)^{-1}] \\
&\qquad = [[x_1(w_2), x_3(w_1)^{-1}], x_4(v)^{-1}] \\
&\qquad = [x_1(w_2), x_3(w_1)^{-1}]^{-1}[x_1(w_2), x_3(w_1)^{-1}]^{x_4(v)^{-1}} \\
&\qquad = [x_1(w_2), x_3(w_1)^{-1}]^{-1}[x_1(w_2)x_2(vw_2)x_3(w_2v), x_3(w_1)^{-1}] .
\end{aligned}$$

If $a_1 \in U_1$, $a_2 \in U_2$ and $a_3, b_3 \in U_3$, then $[a_2, b_3] \in [U_2, U_3] = 1$ and $[a_1, b_3] \in [U_1, U_3] \leq U_2$ (by Theorem 1.4.1(i)), and since $[U_2, U_2U_3] = 1$, we have that $[a_1, b_3]^{a_2a_3} = [a_1, b_3]$. Therefore $[a_1a_2a_3, b_3] = [a_1, b_3][a_3b_3]$. Hence

$$\begin{aligned}
x_3(F(H(w_2,w_1),v)) &= [x_1(w_2), x_3(w_1)^{-1}]^{-1}[x_1(w_2), x_3(w_1)^{-1}] \\
&\qquad \cdot [x_3(w_2v), x_3(w_1)^{-1}] \\
&= [x_3(w_2v), x_3(w_1)^{-1}] \\
&= x_3(\boxminus w_2v \boxplus w_1 \boxplus w_2v \boxminus w_1) ,
\end{aligned}$$

and since $\mathrm{Im}(F) \leq Z(W)$, we have that

$$\begin{aligned}
x_3(F(H(w_2,w_1),v)) &= x_3(\boxminus w_1 \boxminus w_2v \boxplus w_1 \boxplus w_2v) \\
&= x_3([w_1, w_2v]_\boxplus)
\end{aligned}$$

as well, for all $w_1, w_2 \in W$ and all $v \in V$, which proves ($\mathbf{Q}_8$).

We will now apply the Shift Lemma 2.4.3(ii) on the identity

$$[x_1(w), x_4(v)^{-1}] = x_2(vw)x_3(wv) .$$

This gives us the identity

$$[\kappa(x_4(v)), x_3(wv)^{-1}] = x_1(w)x_2(vw) .$$

Note that, by definition, we have $\kappa(x_4(v)) = x_0(v^{-1})$. If we conjugate this identity by $\mu(e_4)$, we thus get, by Theorem 2.4.11, that

$$[x_4(v^{-1}), x_1(wv)^{-1}] = x_3(w(-\epsilon))x_2(-\overline{vw}) .$$

Inverting this identity and replacing w by $\boxminus w$ yields

$$[x_1(wv), x_4(-(v^{-1}))^{-1}] = x_2(\overline{v(\boxminus w)})x_3(w(-\epsilon)) ,$$

for all $w \in W$ and all $v \in V^*$. But on the other hand, we have

$$[x_1(wv), x_4(-(v^{-1}))^{-1}] = x_2(-(v^{-1})(wv))x_3(wv(-(v^{-1}))) ,$$

for all $w \in W$ and all $v \in V^*$. By Theorem 1.4.1(ii), this implies that

$$\begin{aligned} \overline{v(\boxminus w)} &= -(v^{-1})(wv) , \\ w(-\epsilon) &= wv(-(v^{-1})) , \end{aligned}$$

for all $w \in W$ and all $v \in V^*$. If we apply the identity $w(-v) = wv(-\epsilon)$ on the second equality, we can conclude that this is equivalent to

$$\begin{aligned} v^{-1}(wv) &= -\overline{v(\boxminus w)} , \\ wvv^{-1} &= w , \end{aligned}$$

for all $w \in W$ and all $v \in V^*$. So we have proved ($\mathbf{Q}_{16}$) and ($\mathbf{Q}_{15}$).

If we replace v by v^{-1} and w by wv in ($\mathbf{Q}_{16}$), then we get

$$(v^{-1})^{-1}(wvv^{-1}) = -\overline{v^{-1}(\boxminus wv)} ,$$

for all $w \in W$ and all $v \in V^*$. Using ($\mathbf{Q}_{15}$) and ($\mathbf{Q}_{16}$) once again, and using the fact that $-\overline{(-\overline{v})} = v$ for all $v \in V$, we get

$$(v^{-1})^{-1}w = vw ,$$

for all $w \in W$ and all $v \in V^*$. If we choose $w = \delta$, it follows that $(v^{-1})^{-1} = v$, for all $v \in V^*$, which proves ($\mathbf{Q}_{13}$).

If we take $a_i = x_4(v)$ in Lemma 2.4.2(ii), then we get that $\lambda(x_4(-v)) = \kappa(x_4(v))^{-1}$, for all $v \in V^*$. By the definition of v^{-1}, this is equivalent to $x_0((-v)^{-1}) = x_0(-(v^{-1}))$, from which it follows that $(-v)^{-1} = -(v^{-1})$, for all $v \in V^*$.

Similarly, if we choose $a_i = x_1(w)$ in Lemma 2.4.2(ii), then we get that $\lambda(x_1(\boxminus w)) = \kappa(x_1(w))^{-1}$, for all $w \in W^*$. By the definition of κ and λ, this is equivalent to $x_5(\lambda(\boxminus w)) = x_5(\boxminus\kappa(w))$, from which it follows that $\lambda(\boxminus w) = \boxminus\kappa(w)$, for all $w \in W^*$.

If we apply the Shift Lemma 2.4.3(iii) on the identity

$$[x_1(w), x_4(v)^{-1}] = x_2(vw)x_3(wv) ,$$

then we get that

$$[x_2(vw), \lambda(x_1(w))^{-1}] = x_3(wv)x_4(v) ,$$

for all $w \in W^*$ and all $v \in V$. By definition, we have $\lambda(x_1(w)) = x_5(\lambda(w))$. If we conjugate this identity by $\mu(e_1)^{-1}$, we thus get, by Theorem 2.4.11, that

$$[x_4(vw), x_1(\lambda(w))^{-1}] = x_3(wv)x_2(-v) ,$$

for all $w \in W^*$ and all $v \in V$. We can rewrite this identity as

$$[x_1(\boxminus\lambda(w)), x_4(-vw)^{-1}] = x_2(v)x_3(\boxminus wv) ,$$

for all $w \in W^*$ and all $v \in V$. On the other hand, we also have that

$$[x_1(\boxminus\lambda(w)), x_4(-vw)^{-1}] = x_2(-vw(\boxminus\lambda(w)))x_3(\boxminus\lambda(w)(-vw)) ,$$

for all $w \in W^*$ and all $v \in V$. It follows from Theorem 1.4.1(ii) that

$$v = -vw(\boxminus\lambda(w)) ,$$
$$wv = \lambda(w)(-vw) ,$$

for all $w \in W^*$ and all $v \in V$. If we replace w by $\boxminus\lambda(w)$ and v by vw in the second equality, then we get

$$\boxminus\lambda(w)(vw) = \lambda(\boxminus\lambda(w))(-vw(\boxminus\lambda(w))) ,$$

for all $w \in W^*$ and all $v \in V$. If we use these same equalities once again, then we can simplify this to

$$\boxminus w(-v) = \lambda(\boxminus\lambda(w))v ,$$

for all $w \in W^*$ and all $v \in V$. If we choose $v = \epsilon$, then we get $\lambda(\boxminus\lambda(w)) = \boxminus w(-\epsilon)$, for all $w \in W^*$. Since $\lambda(\boxminus w) = \boxminus\kappa(w)$, for all $w \in W^*$, this is the same as $\kappa(\lambda(w)) = w(-\epsilon)$, for all $w \in W^*$, so we have proved ($\mathbf{Q}_{14}$). If we replace w by $\boxminus w$, then we get $\lambda(\kappa(w)) = w(-\epsilon)$ as well. Now, we substitute $\kappa(w)$ for w in the equations

$$v = -vw(\boxminus\lambda(w)) ,$$
$$wv = \lambda(w)(-vw) ;$$

this gives us, using the fact that $\lambda(\kappa(w)) = w(-\epsilon)$, that

$$v = -v\kappa(w)(\boxminus w(-\epsilon)) ,$$
$$\kappa(w)v = w(-\epsilon)(-v\kappa(w)) ,$$

for all $w \in W^*$ and all $v \in V$. It suffices to use ($\mathbf{Q}_6$) and ($\mathbf{Q}_5$) to see that those two equations are equivalent to ($\mathbf{Q}_{18}$) and ($\mathbf{Q}_{19}$), respectively.

If we put $a_2 = x_2(v_2)$ and $a_4 = x_4(v_1)$ in Theorem 2.4.8(ii), then we get

$$[[\lambda(x_4(v_1)), x_2(v_2)^{-1}], x_4(v_1)]_3 = [x_2(v_2), x_4(v_1)]^{-1} ,$$

for all $v_1 \in V^*$ and all $v_2 \in V$. First of all, we have that

$$\begin{aligned}
[\lambda(x_4(v_1)), x_2(v_2)^{-1}] &= [x_0(v_1^{-1}), x_2(v_2)^{-1}] \\
&= [x_4(v_1^{-1}), x_2(-\overline{v_2})^{-1}]^{\mu(e_4)} \\
&= ([x_2(\overline{v_2}), x_4(-v_1^{-1})^{-1}]^{-1})^{\mu(e_4)} \\
&= (x_3(F(\overline{v_2}, -v_1^{-1}))^{-1})^{\mu(e_4)} \\
&= x_3(F(\overline{v_2}, v_1^{-1}))^{\mu(e_4)} \\
&= x_1(F(\overline{v_2}, v_1^{-1})) ,
\end{aligned}$$

for all $v_1 \in V^*$ and all $v_2 \in V$. So it follows from this identity that

$$[x_1(F(\overline{v_2}, v_1^{-1})), x_4(-v_1)^{-1}]_3 = [x_2(v_2), x_4(-v_1)^{-1}]^{-1} ,$$

for all $v_1 \in V^*$ and all $v_2 \in V$, from which it follows that

$$F(\overline{v_2}, v_1^{-1})(-v_1) = F(v_2, v_1) ,$$

for all $v_1 \in V^*$ and all $v_2 \in V$. If we now replace v_1 by $-v_1$, then we get, using the fact that $(-v_1)^{-1} = -(v_1^{-1})$ and ($\mathbf{Q}_{21}$), that ($\mathbf{Q}_{17}$) holds.

If we choose $v_1 = \epsilon$ in ($\mathbf{Q}_{17}$), then we get that $F(\epsilon^{-1}, \overline{v}) = F(\epsilon, v)$, for all $v \in V$. If we put $a_2 = x_2(v)$ and $a_4 = e_4$ in Theorem 2.4.8(i), then we get

$$x_2(v)^{\mu(e_4)} x_2(v)^{-1} = [[\lambda(e_4), x_2(v)^{-1}], e_4]_2 ,$$

for all $v \in V$. We have that

$$\begin{aligned}
[\lambda(e_4), x_2(v)^{-1}] &= x_1(F(\overline{v}, \epsilon^{-1})) \\
&= x_1(F(\epsilon, v)) ,
\end{aligned}$$

for all $v \in V$. Thus we have

$$\begin{aligned}
x_2(v)^{\mu(e_4)} x_2(v)^{-1} &= [[\lambda(e_4), x_2(v)^{-1}], e_4]_2 \\
&= [x_1(F(\epsilon, v)), x_4(-\epsilon)^{-1}]_2 \\
&= x_2(-\epsilon F(\epsilon, v)) ,
\end{aligned}$$

for all $v \in V$. Since $x_2(v)^{\mu(e_4)}x_2(v)^{-1} = x_2(-\overline{v} - v)$, we conclude that

$$\overline{v} = \epsilon F(\epsilon, v) - v ,$$

for all $v \in V$; see page 50.

If we put $a_1 = x_1(w_1)$ and $a_3 = x_3(w_2)$ in Theorem 2.4.8(iii), then we get

$$[x_1(w_1), [x_3(w_2), \kappa(x_1(w_1))]^{-1}]_2 = [x_1(w_1), x_3(w_2)^{-1}]^{-1} ,$$

for all $w_1 \in W$ and all $w_2 \in W^*$. First of all, we have that

$$\begin{aligned}[x_3(w_2), \kappa(x_1(w_1))] &= [x_3(w_2), x_5(\kappa(w_1))] \\ &= [x_3(w_2), x_1(\kappa(w_1))]^{\mu(e_1)} \\ &= ([x_1(\kappa(w_1)), x_3(\boxminus w_2)^{-1}]^{-1})^{\mu(e_1)} \\ &= (x_2(-H(\kappa(w_1), w_2))^{-1})^{\mu(e_1)} \\ &= x_2(H(\kappa(w_1), w_2))^{\mu(e_1)} \\ &= x_4(-H(\kappa(w_1), w_2)) ,\end{aligned}$$

for all $w_1 \in W$ and all $w_2 \in W^*$. So it follows from this identity that

$$[x_1(w_1), x_4(-H(\kappa(w_1), w_2))^{-1}]_2 = [x_1(w_1), x_3(w_2)^{-1}]^{-1} ,$$

for all $w_1 \in W$ and all $w_2 \in W^*$, from which it follows that

$$-H(\kappa(w_1), w_2)w_1 = -H(w_1, w_2) ,$$

for all $w_1 \in W$ and all $w_2 \in W^*$. So we have proved ($\mathbf{Q}_{20}$).

Since we have shown all of the identities ($\mathbf{Q}_1$) – ($\mathbf{Q}_{20}$), we can conclude that every Moufang quadrangle can be obtained from a quadrangular system.

In particular, we are now allowed to use the results of section 2.3 as well. We thus continue to show that the identities ($\mathbf{Q}_{23}$) – ($\mathbf{Q}_{26}$) hold.

In order to show ($\mathbf{Q}_{23}$), we will calculate the expression

$$x_2(v)^{[\mu(x_1(\delta))\mu(x_1(z))]^{\mu(x_3(w))\mu(x_3(\delta))}}$$

with $v \in V$ and $w, z \in W^*$ in two different ways. We have shown in Theorem 2.3.3(iii) that

$$x_3(z)^{\mu(x_1(w))} = x_3(\Pi_w(z))$$

for all $w, z \in W^*$. If we let $\mu(e_4)$ act on both sides of this equality, then it follows by Lemma 2.4.2(iii) and Theorem 2.4.11 that

$$x_1(z)^{\mu(x_3(w(-\epsilon)))} = x_1(\Pi_w(z))$$

and it thus follows by substituting $w(-\epsilon)$ for w and by Lemma 2.4.2(iii) that
$$\mu(x_1(z))^{\mu(x_3(w))} = \mu(x_1(\Pi_{w(-\epsilon)}(z)))$$
for all $w, z \in W^*$. By Lemma 2.2.15(ii), we have $\Pi_{w(-\epsilon)}(z) = \Pi_w(z)$ for all $w, z \in W^*$. Since $\delta \in \mathrm{Rad}(H)$ by ($\mathbf{Q}_9$), it now follows that

$$\mu(x_1(z))^{\mu(x_3(w))\mu(x_3(\delta))} = \mu(x_1(\Pi_w(z)))$$

and hence, since $\Pi_w(\delta) = \delta$,

$$[\mu(x_1(\delta))\mu(x_1(z))]^{\mu(x_3(w))\mu(x_3(\delta))} = \mu(x_1(\delta))\mu(x_1(\Pi_w(z)))$$

for all $w, z \in W^*$. Note that $v\kappa(\delta) = v$ for all $v \in V$. Since we have shown in Theorem 2.3.3 that

$$\begin{aligned} x_2(v)^{\mu(x_1(w))} &= x_4(-v\kappa(w)) \quad \text{and} \\ x_4(v)^{\mu(x_1(w))} &= x_2(vw) \end{aligned}$$

for all $v \in V$ and all $w \in W^*$, it thus follows, by Lemma 2.4.2(i), that

$$\begin{aligned} x_2(v)^{[\mu(x_1(\delta))\mu(x_1(z))]^{\mu(x_3(w))\mu(x_3(\delta))}} &= x_2(v)^{\mu(x_1(\delta))\mu(x_1(\Pi_w(z)))} \\ &= x_4(-v)^{\mu(x_1(\Pi_w(z)))} \\ &= x_2(-v \cdot \Pi_w(z)) \end{aligned}$$

for all $w, z \in W^*$.

On the other hand, if we let $\mu(e_4)$ act on both sides of the identity $x_4(v)^{\mu(x_1(w))} = x_2(vw)$, then we can deduce that

$$\begin{aligned} x_0(v)^{\mu(x_3(w))} &= x_2(-\overline{vw}) \quad \text{and} \\ x_2(v)^{\mu(x_3(w))} &= x_0(\overline{v}\kappa(w)) \end{aligned}$$

for all $v \in V$ and all $w \in W^*$. Hence, by Lemma 2.4.2(i),

$$\begin{aligned} & x_2(v)^{[\mu(x_1(\delta))\mu(x_1(z))]^{\mu(x_3(w))\mu(x_3(\delta))}} \\ &\quad = x_2(v)^{\mu(x_3(\boxminus\delta))\mu(x_3(\boxminus w))\mu(x_1(\delta))\mu(x_1(z))\mu(x_3(w))\mu(x_3(\delta))} \\ &\quad = x_2\big(\overline{\overline{v}(\boxminus w)}\big)^{\mu(x_1(\delta))\mu(x_1(z))\mu(x_3(w))\mu(x_3(\delta))} \\ &\quad = x_2\big(-\overline{\overline{v}(\boxminus w)}z\big)^{\mu(x_3(w))\mu(x_3(\delta))} \\ &\quad = x_2\big(\overline{\overline{\overline{\overline{v}(\boxminus w)}}z\kappa(w)}\big) \end{aligned}$$

for all $w, z \in W^*$. Hence we have shown that ($\mathbf{Q}_{23}$) holds.

The proof of ($\mathbf{Q}_{24}$) follows in a completely similar way by calculating the expression

$$x_3(w)^{[\mu(x_4(\epsilon))\mu(x_4(c))]^{\mu(x_2(v))\mu(x_2(\epsilon))}}$$

with $w \in W$ and $v, c \in V^*$ in two different ways.

We will now show ($\mathbf{Q}_{25}$). Let $c \in V$, $v \in V^*$ and $w \in W^*$ be arbitrary. This time, we will calculate the expression

$$x_2(c)^{[\mu(x_1(\delta))\mu(x_1(w))]^{\mu(x_4(-v))}}$$

in two different ways. By Theorem 2.3.2(ii),

$$x_1(w)^{\mu(x_4(v))} = x_3(w(-v))$$

for all $v \in V^*$ and all $w \in W$, and hence, by Lemma 2.4.2(iii),

$$[\mu(x_1(\delta))\mu(x_1(w))]^{\mu(x_4(-v))} = \mu(x_3(\delta v))\mu(x_3(wv)) \ .$$

It follows that

$$\begin{aligned} x_2(c)^{[\mu(x_1(\delta))\mu(x_1(w))]^{\mu(x_4(-v))}} &= x_2(c)^{\mu(x_3(\delta v))\mu(x_3(wv))} \\ &= x_2(\overline{c} \cdot \kappa(\delta v))^{\mu(x_3(wv))} \\ &= x_2(-\overline{\overline{c} \cdot \kappa(\delta v) \cdot wv}) \ . \end{aligned}$$

On the other hand, we have shown in Theorem 2.3.2(iii) that

$$x_2(u)^{\mu(x_4(v))} = x_2(\pi_v(u))$$

for all $u \in V$ and all $v \in V^*$, and hence, by Lemma 2.4.2(i),

$$\begin{aligned} x_2(c)^{[\mu(x_1(\delta))\mu(x_1(w))]^{\mu(x_4(-v))}} &= x_2(c)^{\mu(x_4(v))\mu(x_1(\delta))\mu(x_1(w))\mu(x_4(-v))} \\ &= x_2(\pi_v(c))^{\mu(x_1(\delta))\mu(x_1(w))\mu(x_4(-v))} \\ &= x_2(-\pi_v(c)w)^{\mu(x_4(-v))} \\ &= x_2(-\pi_{-v}(\pi_v(c)w)) \ . \end{aligned}$$

Since $\pi_{-v}(u) = \pi_v(u)$ for all $u \in V$ and all $v \in V^*$, it follows by comparing these two expressions that

$$\overline{\overline{c} \cdot \kappa(\delta v) \cdot wv} = \pi_v(\pi_v(c)w) \ .$$

If we substitute $\overline{c \cdot \delta v}$ for c in the last identity and apply π_v on both sides, then we get, by Lemma 2.2.23(i), that

$$\pi_v(\overline{c \cdot \delta v \cdot \kappa(\delta v) \cdot wv}) = \pi_v(\overline{c \cdot \delta v})w \ .$$

Since $\delta v \in \mathrm{Rad}(H)$ by $(\mathbf{Q}_9)$ and Lemma 2.2.19, it follows by Lemma 2.2.13(ii) and Lemma 2.2.2(i) that

$$c \cdot \delta v \cdot \kappa(\delta v) = -c \cdot (\boxminus \delta v) \cdot \kappa(\delta v) = c \ ,$$

which completes the proof of $(\mathbf{Q}_{25})$.

The proof of $(\mathbf{Q}_{26})$ follows in a completely similar way by calculating the expression

$$x_3(w)^{[\mu(x_4(\epsilon))\mu(x_4(-v))]^{\mu(x_1(z))}}$$

with $w \in W$, $z \in W^*$ and $v \in V^*$ in two different ways.

This concludes the proof of all of the identities $(\mathbf{Q}_1)$ – $(\mathbf{Q}_{26})$.

2.5 Some Remarks

We start by pointing out that we have really shown that the identities $(\mathbf{Q}_{23})$ – $(\mathbf{Q}_{26})$ follow from the axioms $(\mathbf{Q}_1)$ – $(\mathbf{Q}_{20})$.

Theorem 2.5.1. *Let $\Omega := (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then the identities $(\mathbf{Q}_{23})$ – $(\mathbf{Q}_{26})$ hold, for all $v, c \in V$ and all $w, z \in W$.*

Proof. Let $\Omega := (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then it follows from section 2.3 that we can construct a Moufang quadrangle Γ starting from Ω. In section 2.4, it is shown that every Moufang quadrangle can be constructed from a quadrangular system for which additionally the identities $(\mathbf{Q}_{23})$ – $(\mathbf{Q}_{26})$ hold. In particular, Γ can be constructed from a quadrangular system, which can be chosen to coincide with the quadrangular system Ω that we started with, since the choice of the parametrization of the groups U_1 and U_4 and of the elements ϵ and δ was arbitrary. (Note that the parametrization of the groups U_2 and U_3 and the definition of the maps F and H then automatically coincide by construction.) This shows that Ω is a quadrangular system for which additionally the identities $(\mathbf{Q}_{23})$ – $(\mathbf{Q}_{26})$ hold. Since Ω was arbitrary, these identities hold for every quadrangular system. □

Remark 2.5.2. One might wonder why we pay so much attention to these last four identities $(\mathbf{Q}_{23})$, $(\mathbf{Q}_{24})$, $(\mathbf{Q}_{25})$ and $(\mathbf{Q}_{26})$. The reason is that these identities turn out to be essential for the classification of the quadrangular systems, but still, we are not aware of a direct proof for the fact that they follow from the other axioms.

Remark 2.5.3. Although every quadrangular system gives rise to a Moufang quadrangle and every Moufang quadrangle can be constructed from a quadrangular system, it is *not* true that there is a bijection between the set of classes of isomorphic quadrangular systems and the set of classes of isomorphic Moufang quadrangles. In particular, two non-isomorphic quadrangular systems can give rise to isomorphic Moufang quadrangles. However, two isomorphic quadrangular systems will always give rise to isomorphic Moufang quadrangles.

Remark 2.5.4. We could as well have defined a quadrangular system without axiom ($\mathbf{Q}_{10}$). The reason that we added this axiom has to do with the classification of the so-called *wide* quadrangular systems which are the extension of a quadrangular system of *quadratic form type*. Without axiom ($\mathbf{Q}_{10}$), one would have to define a *translate* of a quadrangular system of type F_4 in order to describe all possible quadrangular systems (up to isomorphism), which is not needed now because of this extra axiom. (See section 2.7.5 for more details.)

On the other hand, if there are no quadrangles of type F_4 involved in a certain application, then it can often be more convenient to drop this axiom ($\mathbf{Q}_{10}$), since it gives more freedom in the choice of the base point $\epsilon \in V^*$. See also Remark 2.4.10 and Remark 2.8.1.

2.6 Examples

We will now present a list of six examples of quadrangular systems. These examples correspond to the six different classes of Moufang quadrangles in [52]. The goal of the next section is to prove that, up to isomorphism, this list is complete.

In each case, we will describe a *parametrization* for the groups V and W, that is, we will describe V and W as groups which are isomorphic to certain other groups $\tilde{V}$ and $\tilde{W}$, respectively; we will denote the isomorphisms from $\tilde{V}$ to V and from $\tilde{W}$ to W by square brackets: $a \in \tilde{V} \mapsto [a] \in V$ and $b \in \tilde{W} \mapsto [b] \in W$.

2.6.1 Quadrangular Systems of Quadratic Form Type

Consider a non-trivial anisotropic quadratic space (K, V_0, q) with base point ϵ. Let V be parametrized by $(V_0, +)$, and let W be parametrized by the additive group of K. We define a map τ_V from $V \times W$ to V and a map τ_W

from $W \times V$ to W as follows:

$$\begin{aligned}\tau_V([v],[t]) &:= [v][t] := [tv]\ ,\\ \tau_W([t],[v]) &:= [t][v] := [tq(v)]\ ,\end{aligned}$$

for all $v \in V_0$ and all $t \in K$. Then $(V, W, \tau_V, \tau_W, [\epsilon], [1])$ is a quadrangular system. One can check that

$$\begin{aligned}F([u],[v]) &= [f(u,v)]\ ,\\ H([s],[t]) &= [0]\ ,\end{aligned}$$

for all $u, v \in V_0$ and all $s, t \in K$, and that

$$\begin{aligned}[v]^{-1} &= [q(v)^{-1}\overline{v}]\ ,\\ \kappa([t]) &= [t^{-1}]\ ,\end{aligned}$$

for all $v \in V_0^*$ and all $t \in K^*$. Note that

$$\overline{[v]} = \epsilon F(\epsilon,[v]) - [v] = [\epsilon][f(\epsilon,v)] - [v] = [\overline{v}]$$

for all $v \in V$.

These are the *quadrangular systems of quadratic form type*. They will be denoted by $\Omega_Q(K, V_0, q)$.

2.6.2 Quadrangular Systems of Involutory Type

Consider an involutory set (K, K_0, σ). Let V be parametrized by the additive group of K, and let W be parametrized by K_0. We define a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W as follows:

$$\begin{aligned}\tau_V([a],[t]) &:= [a][t] := [ta]\ ,\\ \tau_W([t],[a]) &:= [t][a] := [a^\sigma ta]\ ,\end{aligned}$$

for all $a \in K$ and all $t \in K_0$. Then $(V, W, \tau_V, \tau_W, [1], [1])$ is a quadrangular system. One can check that

$$\begin{aligned}F([a],[b]) &= [a^\sigma b + b^\sigma a]\ ,\\ H([s],[t]) &= [0]\ ,\end{aligned}$$

for all $a, b \in K$ and all $s, t \in K_0$, and that

$$\begin{aligned}[a]^{-1} &= [a^{-1}]\ ,\\ \kappa([t]) &= [t^{-1}]\ ,\end{aligned}$$

for all $a \in K^*$ and all $t \in K_0^*$. Note that

$$\overline{[a]} = \epsilon F(\epsilon, [a]) - [a] = [1][a + a^\sigma] - [a] = [a^\sigma]$$

for all $a \in K$.

These are the *quadrangular systems of involutory type*. They will be denoted by $\Omega_I(K, K_0, \sigma)$.

2.6.3 Quadrangular Systems of Indifferent Type

Consider an indifferent set (K, K_0, L_0). Let V be parametrized by L_0, and let W be parametrized by K_0. We define a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W as follows:

$$\tau_V([a], [t]) := [a][t] := [t^2 a] ,$$
$$\tau_W([t], [a]) := [t][a] := [ta] ,$$

for all $a \in L_0$ and all $t \in K_0$. Then $(V, W, \tau_V, \tau_W, [1], [1])$ is a quadrangular system. One can check that

$$F([a], [b]) = [0] ,$$
$$H([s], [t]) = [0] ,$$

for all $a, b \in L_0$ and all $s, t \in K_0$, and that

$$[a]^{-1} = [a^{-1}] ,$$
$$\kappa([t]) = [t^{-1}] ,$$

for all $a \in K^*$ and all $t \in K_0^*$. Note that $\overline{[a]} = [a]$ for all $a \in K$.

These are the *quadrangular systems of indifferent type*. They will be denoted by $\Omega_D(K, K_0, L_0)$.

2.6.4 Quadrangular Systems of Pseudo-quadratic Form Type

Let (K, K_0, σ, V_0, p) be an arbitrary anisotropic pseudo-quadratic space with corresponding skew-hermitian form h, and let the group $(T, \boxplus)$ be as in section 1.9.3. Let V be parametrized by the additive group of K, and let W be parametrized by T. We define a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W as follows:

$$\tau_V([v], [a, t]) := [v][a, t] := [tv] ,$$
$$\tau_W([a, t], [v]) := [a, t][v] := [av, v^\sigma tv] ,$$

for all $v \in K$ and all $(a,t) \in T$. Then $(V, W, \tau_V, \tau_W, [1], [0,1])$ is a quadrangular system. One can check that

$$F([u],[v]) = [0, u^\sigma v + v^\sigma u] \ ,$$
$$H([a,t],[b,s]) = [h(a,b)] \ ,$$

for all $u, v \in K$ and all $(a,t), (b,s) \in T$, and that

$$[v]^{-1} = [v^{-1}] \ ,$$
$$\kappa([a,t]) = [at^{-\sigma}, t^{-\sigma}] \ ,$$

for all $v \in K^*$ and all $(a,t) \in T^*$. Note that

$$\overline{[v]} = [1]F([1],[v]) - [v] = [1][0, v + v^\sigma] - [v] = [v^\sigma]$$

for all $v \in K$.

These are the *quadrangular systems of pseudo-quadratic form type*. They will be denoted by $\Omega_P(K, K_0, \sigma, V_0, p)$.

2.6.5 Quadrangular Systems of Type E_6, E_7 and E_8

Let K be an arbitrary commutative field, let V_0 be a vector space over K, and let q be an anisotropic quadratic form from V_0 to K. Then

- q is a *quadratic form of type* E_6 if and only if $dim_K V_0 = 6$ and q has a norm splitting $q \simeq s_1 N \perp s_2 N \perp s_3 N$.
- q is a *quadratic form of type* E_7 if and only if $dim_K V_0 = 8$ and q has a norm splitting $q \simeq s_1 N \perp \cdots \perp s_4 N$ such that $s_1 s_2 s_3 s_4 \notin N(E)$.
- q is a *quadratic form of type* E_8 if and only if $dim_K V_0 = 12$ and q has a norm splitting $q \simeq s_1 N \perp \cdots \perp s_6 N$ such that $-s_1 s_2 s_3 s_4 s_5 s_6 \in N(E)$.

An anisotropic quadratic space (K, V_0, q) is called of type E_6, E_7 or E_8 if and only if q is a quadratic form of type E_6, E_7 or E_8, respectively.

Theorem 2.6.1. *Let (K, V_0, q) be a quadratic space of type E_k with $k \in \{6,7,8\}$, with base point ϵ. Let T be a norm splitting map of q, and let X_0 be a vector space over K of dimension 2^{k-3}. Then there exists a unique map $(a,v) \mapsto av$ from $X_0 \times V_0$ to X_0 and an element $\xi \in X_0^*$ such that*

$$at = a(t\epsilon) \ ,$$
$$(av)\overline{v} = aq(v) \ ,$$
$$\xi T(v) = (\xi T(\epsilon))v \ ,$$

for all $a \in X_0$, $t \in K$ and $v \in V_0$.

Proof. This follows from [52, (12.56) and (13.11)]. □

From now on, we let T be a fixed arbitrary norm splitting map of q, and we let X_0 be a fixed vector space over K of dimension 2^{k-3}. We apply Theorem 2.6.1 with these choices of T and X_0. Note that ξ is *not* uniquely determined; see [52, (13.12)].

Remark 2.6.2. The first two conditions of Theorem 2.6.1 say that X_0 is a $C(q, \epsilon)$-module, where $C(q, \epsilon)$ is the Clifford algebra of q with base point ϵ as defined on page 19. It actually turns out that the structure of $C(q, \epsilon)$, which is, by Theorem 1.8.6, the same as the structure of the even Clifford algebra $C_0(q)$, plays a crucial role in the understanding of the exceptional Moufang quadrangles of type E_6, E_7, and E_8. In particular, quadratic forms of type E_6, E_7, and E_8 are completely characterized by the structure of their even Clifford algebra only; see chapter 4.

Theorem 2.6.3. *We can choose the norm splitting $(E, \{v_1, \ldots, v_d\})$ in such a way that $v_1 = \epsilon$ (and hence $s_1 = 1$). Furthermore, if $k = 8$, then we can choose it in such a way that $\xi v_2 v_3 v_4 v_5 v_6 = \xi$ as well.*

Proof. This follows from [52, (27.20) and (27.13)]. □

So assume that the norm splitting satisfies the conditions of this Theorem. Then we can now define a subspace M_0 of X_0 as follows.
If $k = 6$, then we set

$$M_0 := \{\xi t v_2 v_3 \mid t \in E\} \,;$$

If $k = 7$, then we set

$$M_0 := \{\xi t_1 v_2 v_3 + \xi t_2 v_1 v_3 + \xi t_3 v_1 v_2 + \xi t v_1 v_2 v_3 \mid t_1, t_2, t_3, t \in E\} \,;$$

If $k = 8$, then we set

$$M_0 := \left\{ \sum_{\substack{i,j \in \{2,\ldots,6\} \\ i<j}} \xi t_{ij} v_i v_j \;\middle|\; t_{ij} \in E \right\} .$$

Theorem 2.6.4. $X_0 = \xi V_0 \oplus M_0$.

Proof. See [52, (13.14)]. □

Theorem 2.6.5. *There is a unique map h from $X_0 \times X_0$ to V_0 which is bilinear over K, such that*

(i) $h(\xi, \xi v) = T(v) - \overline{T}(v)$*, for all* $v \in V_0$*;*

(ii) $h(\xi, a) = 0$, *for all* $a \in M_0$;

(iii) $h(a, b) = -\overline{h(b, a)}$, *for all* $a, b \in X_0$;

(iv) $h(a, bv) = h(b, av) + f(h(a, b), \epsilon)v$, *for all* $a, b \in X_0$ *and all* $v \in V_0$.

Proof. See [52, (13.15)]. □

We now define an element $\zeta \in V_0$ as follows. Note that, if $\mathrm{char}(K) = 2$, then $f(\epsilon, T(\epsilon)) = \alpha \neq 0$ by the definition of T.

$$\zeta := \begin{cases} \epsilon/2 & \text{if } \mathrm{char}(K) \neq 2 \\ T(\epsilon)/f(\epsilon, T(\epsilon)) & \text{if } \mathrm{char}(K) = 2 \end{cases}.$$

Next, let g be the bilinear form from $X_0 \times X_0$ to K given by

$$g(a, b) := f(h(b, a), \zeta)$$

for all $a, b \in X_0$. Set

$$v^* := \begin{cases} 0 & \text{if } \mathrm{char}(K) \neq 2 \\ f(v, \zeta)\epsilon + f(v, \epsilon)\zeta + v & \text{if } \mathrm{char}(K) = 2 \end{cases},$$

for all $v \in V_0$.

Theorem 2.6.6. *There is a unique map θ from $X_0 \times V_0$ to V_0 satisfying the following conditions, for all $a, b \in X_0$ and all $u, v \in V_0$:*

(i) $\theta(\xi, v) = T(v)$;

(ii) $\theta(a + b, v) = \theta(a, v) + \theta(b, v) + h(b, av) - g(a, b)v$;

(iii) $\theta(av, w) = \overline{\theta(a, \bar{w})}q(v) - \overline{\theta(a, v)}f(w, \bar{v}) + f(\theta(a, v), \bar{w})\bar{v} + f(\theta(a, v^*), v)w$.

Proof. See [52, (13.30), (13.31) and (13.37)]. □

Let φ be the map from $X_0 \times V_0$ to K defined as

$$\varphi(a, v) := f(\theta(a, v^*), v),$$

for all $a \in X_0$ and all $v \in V_0$.

Finally, we define a group $(S, \boxplus)$ as $S := X_0 \times K$ where the group action is given by

$$(a, t) \boxplus (b, s) := (a + b, t + s + g(a, b)),$$

for all $(a, t), (b, s) \in S$. One can check that S is indeed a group with neutral element $(0, 0)$, and with the inverse given by $\boxminus(a, t) = (-a, -t + g(a, a))$, for all $(a, t) \in S$.

Let V be parametrized by $(V_0, +)$, and let W be parametrized by S. We define a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W as follows:

$$\begin{aligned}
\tau_V([v],[a,t]) &:= [v][a,t] := [\theta(a,v) + tv] \ , \\
\tau_W([a,t],[v]) &:= [a,t][v] := [av, tq(v) + \varphi(a,v)] \ ,
\end{aligned}$$

for all $v \in V$ and all $(a,t) \in S$. Then $(V, W, \tau_V, \tau_W, [\epsilon], [0,1])$ is a quadrangular system. One can check that

$$\begin{aligned}
F([u],[v]) &= [0, f(u,v)] \ , \\
H([a,t],[b,s]) &= [h(a,b)] \ ,
\end{aligned}$$

for all $u, v \in V$ and all $(a,t), (b,s) \in S$, and that

$$\begin{aligned}
[v]^{-1} &= [q(v)^{-1}\overline{v}] \ , \\
\kappa([a,t]) &= \left[\frac{a\theta(a,\epsilon) + ta}{q(\theta(a,\epsilon) + t\epsilon)}, \frac{t}{q(\theta(a,\epsilon) + t\epsilon)}\right] \ ,
\end{aligned}$$

for all $v \in K^*$ and all $(a,t) \in S^*$.

Remark 2.6.7. It is not obvious at all to verify that this is a quadrangular system. Quite a lot of identities involving these functions h, g, θ and φ are needed. We will omit these calculations, since our main interest here is to give a *classification*, and not to prove *existence*. However, see [52, Chapter 13 and (32.2)] for more details about these identities.

These are the *quadrangular systems of type E_6, E_7 and E_8*. They will be denoted by $\Omega_E(K, V_0, q)$.

2.6.6 Quadrangular Systems of Type F_4

Consider an anisotropic quadratic space (K, V_0, q) with base point ϵ. Assume that $\mathrm{char}(K) = 2$ and that the quadratic form has non-trivial radical $R := \mathrm{Rad}(f) = \{v \in V_0 \mid f(v, V_0) = 0\} \neq 0$. Then this quadratic space is said to be *of type F_4* if and only if $L := q(R)$ is a subfield of K, and there is a complement S of R in V_0 such that the restriction of q to the subspace S has a norm splitting $(E, \{v_1, v_2\})$ with constants $s_1, s_2 \in K^*$ such that $s_1 s_2 \in L^*$.

From now on, we will assume that (K, V_0, q) is of type F_4. Since $t^2 = q(t\epsilon) \in q(R) = L$ for all $t \in K$, we have that $K^2 \subseteq L \subseteq K$. Denote the restriction of q to S by q_1. Denote the norm of the extension E/K by N, and denote the non-trivial element of $\mathrm{Gal}(E/K)$ by $u \mapsto \overline{u}$ (not to be confused with the map $v \mapsto \overline{v}$ in the definition of a quadrangular system). Set $B_0 := E \oplus E$. Then B_0 is a 4-dimensional vector space over K which can be identified with S by the relation

$$(u, v) \in B_0 \longleftrightarrow uv_1 + vv_2 \in S .$$

In particular, we will write $q_1(u, v) = s_1 N(u) + s_2 N(v)$ for all $(u, v) \in B_0$.

Next, we define a commutative field $D := E^2 L = \{u^2 s \mid u \in E, s \in L\}$. Then $E^2 \subseteq D \subseteq E$, D/L is a separable quadratic extension, and $D \cap K = L$. The non-trivial element of $\mathrm{Gal}(D/L)$ is precisely the restriction of the map $u \mapsto \overline{u}$ to D; hence we will also denote it by $x \mapsto \overline{x}$. Also, the norm of D is precisely the restriction of N to D, and so we will denote it by N as well. Now set $A_0 := D \oplus D$; then A_0 is a 4-dimensional vector space over L. Observe that both $s_1^{-1} s_2$ and $s_1^{-3} s_2$ are elements of L. We now define a quadratic form q_2 on A_0 given by

$$q_2(x, y) := s_1^{-1} s_2 N(x) + s_1^{-3} s_2 N(y)$$

for all $(x, y) \in A_0$. If we set $\alpha := s_1^{-1} s_2 \in L$ and $\beta := s_1^{-1} \in K$, then we have

$$\begin{aligned} q_1(u, v) &= \beta^{-1} \cdot (N(u) + \alpha N(v)) && \text{for all } (u, v) \in B_0 . \\ q_2(x, y) &= \alpha \cdot (N(x) + \beta^2 N(y)) && \text{for all } (x, y) \in A_0 . \end{aligned}$$

We will denote the bilinear forms corresponding to q_1 and q_2 by f_1 and f_2, respectively.

Theorem 2.6.8. *For all $(u, v) \in B_0$ and all $(x, y) \in A_0$ we have:*

(i) $q_1(u, v) \in L \iff (u, v) = (0, 0)$*;*

(ii) $q_2(x, y) \in K^2 \iff (x, y) = (0, 0)$*;*

(iii) $\alpha \in L \setminus K^2$*;*

(iv) $\beta \in K \setminus L$.

Proof. See [52, (14.8)]. □

Note that it follows from (iii) and (iv) of this Theorem that $K^2 \subset L \subset K$. In particular, K is *not* perfect.

Since $L \subseteq K$, we can consider K as a (left) vector space over L by the trivial scalar multiplication $s \cdot t := st$ for all $s \in L$ and all $t \in K$. Since $K^2 \subseteq L$ and $\mathrm{char}(K) = 2$, we can also consider L as a (left) vector space over K by the scalar multiplication $t * s := t^2 s$ for all $t \in K$ and all $s \in L$.

One can check that in this sense, q is a vector space isomorphism from R to $L = q(R)$.

From now on, we will identify R with L via q, and we still identify S with $B_0 = E \oplus E$. Combining those two identifications, we have actually identified V_0 with $B_0 \oplus L$. Then $\epsilon = (0, 1)$, and we have $q(b, s) = q_1(b) + s$, for all $(b, s) \in V_0$.

Now set $W_0 := A_0 \oplus K$. Then W_0 is a vector space over L, and we can define a quadratic form $\hat{q}$ from W_0 to f given by $\hat{q}(a, t) = q_2(a) + t^2$ for all $(a, t) \in W_0$. It follows from Theorem 2.6.8(ii) that $\hat{q}$ is anisotropic as well. One can actually check that $(L, W_0, \hat{q})$ is again a quadratic form of type F_4.

Finally, we define a map Θ from $A_0 \oplus B_0$ to B_0, a map Υ from $A_0 \oplus B_0$ to A_0, a map ν from $A_0 \oplus B_0$ to K, and a map ψ from $A_0 \oplus B_0$ to L as follows.

$$\begin{aligned}
\Theta((x, y), (u, v)) &:= (\alpha \cdot (\bar{x}v + \beta y\bar{v}), xu + \beta y\bar{u}) \ , \\
\Upsilon((x, y), (u, v)) &:= (y\bar{u}^2 + \alpha\bar{y}v^2, \beta^{-2} \cdot (xu^2 + \alpha\bar{x}v^2)) \ , \\
\nu((x, y), (u, v)) &:= \alpha \cdot (\beta^{-1} \cdot (xu\bar{v} + \bar{x}\bar{u}v) + y\bar{u}\bar{v} + \bar{y}uv) \ , \\
\psi((x, y), (u, v)) &:= \alpha \cdot (x\bar{y}u^2 + \bar{x}y\bar{u}^2 + \alpha \cdot (xy\bar{v}^2 + \bar{x}\bar{y}v^2)) \ ,
\end{aligned}$$

for all $(x, y) \in A_0 = D \oplus D$ and all $(u, v) \in B_0 = E \oplus E$.

Let V be parametrized by $(V_0, +)$, and let W be parametrized by $(W_0, +)$. We define a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W as follows:

$$\begin{aligned}
\tau_V([b, s], [a, t]) &:= [b, s][a, t] := [\Theta(a, b) + tb, \hat{q}(a, t)s + \psi(a, b)] \ , \\
\tau_W([a, t], [b, s]) &:= [a, t][b, s] := [\Upsilon(a, b) + sa, q(b, s)t + \nu(a, b)] \ ,
\end{aligned}$$

for all $(b, s) \in V_0$ and all $(a, t) \in W_0$. Then $(V, W, \tau_V, \tau_W, [0, 1], [0, 1])$ is a quadrangular system. One can check that

$$\begin{aligned}
F([b, s], [b', s']) &= [0, f_1(b, b')] \ , \\
H([a, t], [a', t']) &= [0, f_2(a, a')] \ ,
\end{aligned}$$

for all $(b, s), (b', s') \in V_0$ and all $(a, t), (a', t') \in W_0$, and that

$$\begin{aligned}
[b, s]^{-1} &= [q(b, s)^{-1}b, q(b, s)^{-2}s] \ , \\
\kappa([a, t]) &= [\hat{q}(a, t)^{-1}a, \hat{q}(a, t)^{-1}t] \ ,
\end{aligned}$$

for all $(b, s) \in V_0^*$ and all $(a, t) \in W_0^*$.

Remark 2.6.9. It would be a very tedious job to check that this is indeed a quadrangular system by only using the definitions of the different functions involved. However, it is not very hard to prove the following list of twelve identities, after which the verification of the axioms for the quadrangular systems is straightforward.

Theorem 2.6.10. *For all $a, a' \in A_0$ and all $b, b' \in B_0$, we have that*

(i) $\nu(a, b + b') = \nu(a, b) + \nu(a, b') + f_1(\Theta(a, b), b')$;

(ii) $\psi(a + a', b) = \psi(a, b) + \psi(a', b) + f_2(\Upsilon(a, b), a')$;

(iii) $\Upsilon(\Upsilon(a, b), b) = q_1(b)^2 a$;

(iv) $\Theta(a, \Theta(a, b)) = q_2(a) b$;

(v) $\Theta(\Upsilon(a, b), b) + b\nu(a, b) = q_1(b)\Theta(a, b)$;

(vi) $\Upsilon(a, \Theta(a, b)) + a\psi(a, b) = q_2(a)\Upsilon(a, b)$;

(vii) $\nu(\Upsilon(a, b), b) = q_1(b)\nu(a, b)$;

(viii) $\psi(a, \Theta(a, b)) = q_2(a)\psi(a, b)$;

(ix) $\psi(\Upsilon(a, b), b) = q_1(b)^2\psi(a, b)$;

(x) $\nu(a, \Theta(a, b)) = q_2(a)\nu(a, b)$;

(xi) $q_1(\Theta(a, b)) = q_1(b)q_2(a) + \psi(a, b)$;

(xii) $q_2(\Upsilon(a, b)) = q_1(b)^2 q_2(a) + \nu(a, b)^2$.

These are the *quadrangular systems of type F_4*. They will be denoted by $\Omega_F(K, V_0, q)$.

Remark 2.6.11. Although it can be very useful to have these explicit formulas to calculate with, this description is not very clarifying. The description in terms of the quadrangular systems ought to give more insight in the structure of these F_4-quadrangles; see section 2.8.3 and chapter 3.

2.7 The Classification

We will now start the classification of the quadrangular systems. We start with some definitions.

Definition 2.7.1. A quadrangular system $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ is called *indifferent* if $F \equiv 0$ and $H \equiv 0$, *reduced* if $F \not\equiv 0$ and $H \equiv 0$ and *wide* if $F \not\equiv 0$ and $H \not\equiv 0$.

Remark 2.7.2. We will prove that if $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ is a quadrangular system with $F \equiv 0$ and $H \not\equiv 0$, then $\Omega^* := (W, V, \tau_W, \tau_V, \delta, \epsilon)$ is a reduced quadrangular system; see Theorem 2.7.13.

Remark 2.7.3. Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. If $X \subseteq V$ and $Y \subseteq W$, then the restriction of τ_V to $X \times Y$ and the restriction of τ_W to $Y \times X$ will also be denoted by τ_V and τ_W, respectively.

Definition 2.7.4. Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a wide quadrangular system. Set $Y := \mathrm{Rad}(H)$. We will show below that $\Gamma := (V, Y, \tau_V, \tau_W, \epsilon, \delta)$ is a reduced quadrangular system; see Theorem 2.7.14. We then say that Ω is an *extension* of Γ.

Definition 2.7.5. Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a reduced quadrangular system. Then Ω is said to be *normal* if and only if for all $w_1, w_2, \ldots, w_i \in W$, there exists a $w \in W$ such that $\epsilon w_1 w_2 \ldots w_i = \epsilon w$.

Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be an arbitrary quadrangular system. The classification will be divided up into the following five theorems.

Theorem 2.7.6. *If Ω is reduced but not normal, then $\Omega \cong \Omega_I(K, K_0, \sigma)$ for some involutory set (K, K_0, σ) such that $\sigma \neq 1$ and K is generated by K_0 as a ring.*

Theorem 2.7.7. *If Ω is normal, then $\Omega \cong \Omega_Q(K, V_0, q)$ for some anisotropic quadratic space (K, V_0, q).*

Theorem 2.7.8. *If Ω is indifferent, then $\Omega \cong \Omega_D(K, K_0, L_0)$ for some indifferent set (K, K_0, L_0).*

Theorem 2.7.9. *If Ω is an extension of the reduced quadrangular system $\Gamma = \Omega_I(K, K_0, \sigma)$ for some involutory set (K, K_0, σ) such that $\sigma \neq 1$ and K is generated by K_0 as a ring, then $\Omega \cong \Omega_P(K, K_0, \sigma, V_0, p)$ for some anisotropic pseudo-quadratic space (K, K_0, σ, V_0, p).*

Theorem 2.7.10. *If Ω is an extension of the reduced quadrangular system $\Gamma = \Omega_Q(K, V_0, q)$ for some anisotropic quadratic space (K, V_0, q), then one of the following holds:*

- *There exists*

  (a) *a multiplication on V_0 making the vector space V_0 into an algebra over K such that either V_0 is a field and V_0/K is a separable quadratic extension with norm q or V_0 is a quaternion division algebra over K with norm q,*
  (b) *an involution σ of V_0 (which is the unique non-trivial element of $\mathrm{Gal}(V_0/K)$ if $\dim_K V_0 = 2$ and which is the standard involution of V_0 if $\dim_K V_0 = 4$),*
  (c) *a non-trivial right vector space X over V_0,*
  (d) *a pseudo-quadratic form π on X,*

  such that (V_0, K, σ, X, π) is an anisotropic pseudo-quadratic space, $\Gamma \cong \Omega_I(V_0, K, \sigma)$ and $\Omega \cong \Omega_P(V_0, K, \sigma, X, \pi)$.

- (K, V_0, q) *is a quadratic space of type* E_6, E_7 *or* E_8, *and* $\Omega \cong \Omega_E(K, V_0, q)$.
- (K, V_0, q) *is a quadratic space of type* F_4, *and* $\Omega \cong \Omega_F(K, V_0, q)$.

We will now prove the two theorems which we mentioned in the above remarks. But first, we make an easy but useful observation.

Lemma 2.7.11. *If* $\mathrm{Rad}(F) \neq 0$, *then* W *is abelian, and all elements of* V *and* W *have order* 1 *or* 2. *Furthermore,* $\pi_r(v) = v$ *for all* $r \in \mathrm{Rad}(F)^*$ *and all* $v \in V$, *and* $\overline{v} = v$ *for all* $v \in V$.

Proof. Let r be an arbitrary non-zero element of $\mathrm{Rad}(F)$. If we substitute r for v in $(\mathbf{Q}_8)$, then we get that $[w_1, w_2 r]_\boxplus = 0$ for all $w_1, w_2 \in W$. Substituting $w_2 r^{-1}$ for w_2 shows, by $(\mathbf{Q}_{15})$, that W is abelian.

If we substitute r for v in 2.2.13(i), then we get that

$$w(-r) = F(rw, r) \boxminus wr = \boxminus wr$$

for all $w \in W$. By $(\mathbf{Q}_5)$, this implies that $w(-\epsilon)r = \boxminus wr$, and therefore $w(-\epsilon) = \boxminus w$ for all $w \in W$. It thus follows from $(\mathbf{Q}_6)$ that $v(\boxminus w) = vw$ for all $v \in V$ and all $w \in W$. In particular, $v(\boxminus\delta) = v$. On the other hand, it follows from 2.2.13(ii) that $v(\boxminus\delta) = H(\delta, \delta v) - v\delta = -v$, for all $v \in V$. Hence $v = -v$ for all $v \in V$, so every element of V has order 1 or 2.

In particular, $\epsilon = -\epsilon$, hence $w = w(-\epsilon) = \boxminus w$ for all $w \in W$, that is, every element of W has order 1 or 2.

Finally, $\pi_r(v) = v + \overline{r^{-1}F(r, v)} = v$ for all $r \in \mathrm{Rad}(F)^*$ and all $v \in V$. Since it follows from $(\mathbf{Q}_{10})$ that $\epsilon \in \mathrm{Rad}(F)$, we have in particular that $\overline{v} = \pi_\epsilon(v) = v$ for all $v \in V$. □

Remark 2.7.12. Apart from the last statement, we have avoided to use $(\mathbf{Q}_{10})$. We thereby want to stress the fact that this axiom is not essential, and is really only needed to simplify the list of the wide quadrangular systems which have $\mathrm{Rad}(F) \neq 0$. See Remark 2.5.4.

Theorem 2.7.13. *Let* $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ *be a quadrangular system with* $F \equiv 0$ *and* $H \not\equiv 0$. *Then* $\Omega^* := (W, V, \tau_W, \tau_V, \delta, \epsilon)$ *is a reduced quadrangular system.*

Proof. Since $F \equiv 0$, it follows from Lemma 2.7.11 that W is abelian and that all elements of V and W have order 1 or 2. In particular, we will write $+$ in place of $\boxplus$ and $\boxminus$. We define $\overline{w} := \delta H(\delta, w) + w$, for all $w \in W$. Then it follows from $(\mathbf{Q}_9)$ that $\overline{w} = w$, for all $w \in W$. We also set $w^{-1} := \kappa(w)$ for all $w \in W^*$ and $\kappa(v) := v^{-1}$ for all $v \in V^*$. Let $F^* \equiv H$ and $H^* \equiv F \equiv 0$.

We will denote the axioms that we have to prove for Ω^* by $(\mathbf{Q}_i)^*$.

Since all elements of V and W have order 1 or 2, the axioms $(\mathbf{Q}_5)^*$ and $(\mathbf{Q}_6)^*$ are trivial. Note that $\overline{v} = v$ for all $v \in V$ and $\overline{w} = w$ for all $w \in W$.

We now prove the remaining axioms. We observe that $(\mathbf{Q}_1)^* \equiv (\mathbf{Q}_2)$, $(\mathbf{Q}_2)^* \equiv (\mathbf{Q}_1)$, $(\mathbf{Q}_3)^* \equiv (\mathbf{Q}_4)$, $(\mathbf{Q}_4)^* \equiv (\mathbf{Q}_3)$, $(\mathbf{Q}_{13})^* \equiv (\mathbf{Q}_{14})$, $(\mathbf{Q}_{14})^* \equiv (\mathbf{Q}_{13})$, $(\mathbf{Q}_{17})^* \equiv (\mathbf{Q}_{20})$ and $(\mathbf{Q}_{20})^* \equiv (\mathbf{Q}_{17})$. It follows from $F \equiv 0$ that $\mathrm{Rad}(F) = V$ and hence $\mathrm{Im}(H) \subseteq \mathrm{Rad}(F) \ni \epsilon$; this shows $(\mathbf{Q}_7)^*$ and $(\mathbf{Q}_9)^*$. Now $(\mathbf{Q}_8)^*$ follows from the fact that V is abelian and that $F \equiv 0$; $(\mathbf{Q}_{10})^*$ follows from $(\mathbf{Q}_9)$. By $(\mathbf{Q}_{22})$, $H(w_1, w_2) = H(w_2, w_1)$ for all $w_1, w_2 \in W$. Since W is abelian, it follows from $(\mathbf{Q}_{12})$ that $v(w_1 + w_2) = v(w_2 + w_1) = vw_2 + vw_1 + H(w_1, w_2v) = vw_1 + vw_2 + H(w_2v, w_1)$ for all $v \in V$ and all $w_1, w_2 \in W$. This proves $(\mathbf{Q}_{11})^*$. Vice versa, it follows from $(\mathbf{Q}_{11})$ that $w(v_1 + v_2) = wv_1 + wv_2$ for all $w \in W$ and all $v_1, v_2 \in V$, which proves $(\mathbf{Q}_{12})^*$. By $(\mathbf{Q}_{13})$, we have that $(\mathbf{Q}_{15})$ is equivalent to $wv^{-1}v = w$. It follows that $(\mathbf{Q}_{15})^* \equiv (\mathbf{Q}_{18})$ and $(\mathbf{Q}_{18})^* \equiv (\mathbf{Q}_{15})$. Again by $(\mathbf{Q}_{13})$, we have that $(\mathbf{Q}_{16})$ is equivalent to $v(wv^{-1}) = v^{-1}w$. Hence $(\mathbf{Q}_{16})^* \equiv (\mathbf{Q}_{19})$ and $(\mathbf{Q}_{19})^* \equiv (\mathbf{Q}_{16})$. It follows that Ω^* is a quadrangular system, which is reduced since $H^* \equiv 0$ and $F^* \not\equiv 0$. □

Theorem 2.7.14. *Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a wide quadrangular system. Set $Y := \mathrm{Rad}(H)$. Then $\Gamma := (V, Y, \tau_V, \tau_W, \epsilon, \delta)$ is a reduced quadrangular system; see Remark 2.7.3.*

Proof. First of all, we observe that Y is a subgroup of W: if $w_1, w_2 \in Y$, then $H(w_1 \boxplus w_2, w) = H(w_1, w) + H(w_2, w) = 0$ for all $w \in W$, so $w_1 \boxplus w_2 \in Y$ as well. It only remains to show that $\tau_W(Y \times V) \subseteq Y$, $F(V, V) \subseteq Y$ and $\kappa(Y^*) \subseteq Y$. So let w be an arbitrary element of Y, and let v be an arbitrary element of V. Then $[w, w_2]_\boxplus = 0$ for all $w_2 \in W$ by $(\mathbf{Q}_8)$, and therefore $v(w \boxplus w_2) = v(w_2 \boxplus w)$. It follows from $(\mathbf{Q}_{12})$ that $H(w_2, wv) = H(w, w_2v) = 0$ for all $w_2 \in W$. By $(\mathbf{Q}_{22})$, this implies that $H(wv, w_2) = 0$ for all $w_2 \in W$, hence $wv \in Y$. So we have proved that $\tau_W(Y \times V) = Y \cdot V \subseteq Y$.

It follows from $(\mathbf{Q}_7)$ that $F(V, V) \subseteq Y$. Now let w be an arbitrary element of Y^*. Substituting $-\epsilon$ for v in 2.2.2(ii) yields $\kappa(w)(-\epsilon(\boxminus w)) = w$, hence $\kappa(w) = w(-\epsilon(\boxminus w))^{-1} \in Y \cdot V \subseteq Y$. So Γ is a quadrangular system, which is reduced since H restricted to $Y \times Y$ is identically zero. □

2.7.1 Quadrangular Systems of Involutory Type

Our goal in this section is to classify the quadrangular systems which are reduced but not normal.

Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. For the moment, we only assume that $H \equiv 0$, so Ω is reduced or indifferent.

Since $H \equiv 0$, it follows from ($\mathbf{Q}_8$) that W is abelian. We will write $+$ and $-$ in place of $\boxplus$ and $\boxminus$, respectively. It follows from 2.2.13(ii) that $v(-w) = -vw$ for all $v \in V$ and all $w \in W$. If we replace w_2 by $-w_2$ in ($\mathbf{Q}_{12}$), we get $v(w_1 - w_2) = vw_1 - vw_2$, for all $v \in V$ and all $w_1, w_2 \in W$. In particular, it follows from $\epsilon w_1 = \epsilon w_2$ that $w_1 = w_2$, by 2.2.3(ii).

By ($\mathbf{Q}_{18}$), we have $\epsilon\kappa(w)w = \epsilon$ for all $w \in W$. If we replace w by $-w$, we get that $\epsilon(-\kappa(-w))w = \epsilon$ for all $w \in W$. Hence $\epsilon\kappa(w)w = \epsilon(-\kappa(-w))w$, so $\epsilon\kappa(w) = \epsilon(-\kappa(-w))$ by 2.2.5(ii) and hence $\kappa(w) = -\kappa(-w)$ for all $w \in W$ by the previous paragraph. Moreover, by ($\mathbf{Q}_6$) and the result of the previous paragraph, we have that $w(-\epsilon) = w$ for all $w \in W$. It now follows from ($\mathbf{Q}_{14}$) that $\kappa(\kappa(w)) = w$, for all $w \in W$. Hence we will write w^{-1} in place of $\kappa(w)$, for all $w \in W$. Note that it follows from 2.2.2(i) that $vww^{-1} = v$, for all $v \in V$ and all $w \in W^*$.

For all $w_1, w_2, \ldots, w_n \in W^*$, let $m = w_1 \bullet w_2 \bullet \cdots \bullet w_n$ be the automorphism of V which maps v to $vw_1w_2\ldots w_n$ for all $v \in V$; see Theorem 2.2.6(i). Let M be the set of all such automorphisms. Then $(M, \bullet)$ is a group with neutral element δ. We denote the action of an element $m \in M$ by right juxtaposition, i.e. $vm = vw_1w_2\ldots w_n$. Let K be the set of homomorphisms from V to itself (additively) generated by a finite number of elements of M. We write $k = m_1 + \cdots + m_\ell$, where $m_1, \ldots, m_\ell \in M$. Again, we denote the action of an element $k \in K$ by right juxtaposition, so we have $vk = vm_1 + \cdots + vm_\ell$. Then K with this $+$ as addition and with $\bullet$ as multiplication is a ring with multiplicative identity δ. Note that, by ($\mathbf{Q}_{12}$), $v(w_1 + w_2) = vw_1 + vw_2$, hence the notation $w_1 + w_2$ is unambiguous. Let σ be the automorphism of M which maps $m = w_1 \bullet \cdots \bullet w_n$ to $m^\sigma := w_n \bullet \cdots \bullet w_1$. We extend σ to K by setting $k^\sigma := m_1^\sigma + \cdots + m_\ell^\sigma$ for all $k = m_1 + \cdots + m_\ell \in K$. Let $E := \epsilon K$.

Lemma 2.7.15. *$w \cdot \epsilon m \cdot v = w \cdot vm$ for all $v \in V$, $w \in W$ and $m \in M$.*

Proof. Let $m = w_1 \bullet \cdots \bullet w_n$ be an arbitrary element of M, so $w_1, \ldots, w_n$ are elements of W^*. We will show the lemma by induction on n.

Note that $\Pi_{\boxminus z} \equiv \mathrm{id}_W$ since $H \equiv 0$; hence by ($\mathbf{Q}_{26}$), the lemma holds for $n = 1$. Assume that $w \cdot \epsilon w_1 \ldots w_{n-1} \cdot v = w \cdot vw_1 \ldots w_{n-1}$ for all $w \in W$ and all $v \in V$. Then, by repeated use of ($\mathbf{Q}_{26}$),

$$\begin{aligned} w \cdot \epsilon w_1 \ldots w_n \cdot v &= w \cdot (\epsilon w_1 \ldots w_{n-1} \cdot w_n) \cdot v \\ &= w \cdot (\epsilon w_n) \cdot \epsilon w_1 \ldots w_{n-1} \cdot v \\ &= w \cdot (\epsilon w_n) \cdot vw_1 \ldots w_{n-1} \\ &= w \cdot (vw_1 \ldots w_{n-1} \cdot w_n) \\ &= w \cdot vw_1 \ldots w_n \end{aligned}$$

for all $w \in W$ and all $v \in V$, and we are done. $\square$

Lemma 2.7.16. *For all $v \in V$ and all $w, z \in W$, we have that $\overline{\overline{v}wz} = \overline{\overline{vz}w}$.*

Proof. Let $v \in V$ and $w, z \in W$. We may assume that $w \neq 0$. Note that $\Pi_w(z) = z$ since $H \equiv 0$. Since $v(-w) = -vw$, it follows from $(\mathbf{Q}_{23})$ that

$$vz = \overline{\overline{\overline{\overline{v}w}z}w^{-1}} \ .$$

Since $(w^{-1})^{-1} = w$ and by 2.2.12, it follows from this that $\overline{\overline{vz}w} = \overline{\overline{v}wz}$. $\square$

Lemma 2.7.17. *For all $v \in V$, all $w \in W$ and all $m \in M$, we have that $\overline{\overline{v}wm} = \overline{\overline{vm}w}$ and $\overline{\overline{v}mw} = \overline{\overline{vw}m}$.*

Proof. Let $m = w_1 \bullet \cdots \bullet w_n$ be an arbitrary element of M, so $w_1, \ldots, w_n$ are elements of W^*. We will prove by induction on n that

$$\overline{\overline{v}w}w_1 \ldots w_n = \overline{\overline{vw_1 \ldots w_n}w} \ .$$

For $n = 1$, this was shown in Lemma 2.7.16. Assume that we have proved the current lemma for $n-1$. Then

$$\begin{aligned} \overline{\overline{v}w}w_1 \ldots w_n &= \overline{\overline{v}w}w_1 \ldots w_{n-1} \cdot w_n \\ &= \overline{\overline{vw_1 \ldots w_{n-1}}ww_n} \\ &= \overline{\overline{vw_1 \ldots w_n}w} \ , \end{aligned}$$

again by Lemma 2.7.16 with $vw_1 \ldots w_{n-1}$ in place of v. This proves the first identity; the second then follows from the first by substituting $\overline{v}$ for v. $\square$

Lemma 2.7.18. *For all $v \in V$ and all $k_1, k_2 \in m$, we have that $\overline{\overline{v}k_1}k_2 = \overline{\overline{vk_2}k_1}$.*

Proof. Since both sides are additive in k_1 and k_2, it suffices to show this for $k_1 = m_1 \in M$ and $k_2 = m_2 \in M$. Let $m_2 = z_1 \bullet \cdots \bullet z_n$ for some $z_1, \ldots, z_n \in W^*$. By repeated use of Lemma 2.7.17, we have that

$$\begin{aligned} \overline{\overline{v}m_1}m_2 &= \overline{\overline{v}m_1}z_1 \ldots z_n \\ &= \overline{\overline{vz_1}m_1}z_2 \ldots z_n \\ &= \ldots \\ &= \overline{\overline{vz_1 \ldots z_n}m_1} \\ &= \overline{\overline{vm_2}m_1} \ , \end{aligned}$$

which proves the lemma. $\square$

Theorem 2.7.19. *$\overline{\epsilon k} = \epsilon k^\sigma$, for all $k \in K$.*

Proof. It is sufficient to prove this for $k \in M$. Let $k = w_1 \bullet \cdots \bullet w_n$ be an arbitrary element of M, so $w_1, \dots, w_n \in W^*$. We will prove by induction on n that $\overline{\epsilon w_1 \dots w_n} = \epsilon w_n \dots w_1$.

First assume that $n = 1$. Since $v(-w) = -vw$ for all $v \in V$ and all $w \in W$, it follows from 2.2.21 that $\overline{\epsilon w_1} = -\pi_\epsilon(\epsilon w_1) = -\epsilon(-w_1) = \epsilon w_1$.

Now assume that we have proved that $\overline{\epsilon w_1 \dots w_{n-1}} = \epsilon w_{n-1} \dots w_1$, for all $w_1, \dots, w_{n-1} \in W^*$. Then it follows from 2.2.12 and Lemma 2.7.17 that

$$\begin{aligned} \overline{\epsilon w_1 \dots w_n} &= \overline{\overline{\overline{\epsilon w_1 \dots w_{n-1}}} w_n} \\ &= \overline{\overline{\epsilon w_{n-1} \dots w_1} w_n} \\ &= \overline{\overline{\epsilon w_n}} w_{n-1} \dots w_1 \\ &= \epsilon w_n w_{n-1} \dots w_1 \ , \end{aligned}$$

since $\overline{\overline{\epsilon} w_n} = \overline{\epsilon w_n} = \epsilon w_n$. □

Theorem 2.7.20. *$\Delta := (E, W, \tau_V, \tau_W, \epsilon, \delta)$ is a quadrangular system; see Remark 2.7.3.*

Proof. First of all, we observe that $E = \epsilon K$ is a subgroup of V, since $\epsilon k_1 + \epsilon k_2 = \epsilon(k_1 + k_2)$ for all $k_1, k_2 \in K$. It only remains to show that $\tau_V(E \times W) \subseteq E$, $H(W,W) \subseteq E$ and $(E^*)^{-1} \subseteq E$. Since $K \bullet W = K$, we have that $\tau_V(E \times W) = \epsilon K \cdot W = \epsilon(K \bullet W) = \epsilon K = E$. Since $H \equiv 0$, it is obvious that $H(W,W) \subseteq E$. Finally, if we substitute δ for w in $(\mathbf{Q}_{16})$ and apply the fact that $vww^{-1} = v$, we get that $v^{-1} = \overline{v}(\delta v)^{-1}$ for all $v \in V^*$. In particular, we get that $(\epsilon k)^{-1} = \overline{\epsilon k}(\delta \cdot \epsilon k)^{-1} = \epsilon(k^\sigma \bullet (\delta \cdot \epsilon k)^{-1}) \in \epsilon K$ for all $k \in K^*$, where we have used Theorem 2.7.19. Thus $\Delta := (E, W, \tau_V, \tau_W, \epsilon, \delta)$ is a quadrangular system. □

Lemma 2.7.21. *If $vk = 0$ for some $v \in V$ and some $k \in K$, then $vk_2k = 0$ for all $k_2 \in K$.*

Proof. We may assume that $v \neq 0$. We will first show the lemma for $k_2 = w \in W$. Since $v(-w) = -vw$, it follows from 2.2.20 that $vw = \overline{\overline{v}m}$ where $m = \delta\overline{v^{-1}} \bullet \underline{wv} \in M$. By Lemma 2.7.18, it follows from $vk = 0$ that $vwk = \overline{\overline{v}m}k = \overline{\overline{v}k}m = 0$, which proves the lemma in this case.

Now let $k_2 = m = w_1 \bullet \cdots \bullet w_n$ be an arbitrary element of M, then it follows by induction on n that $vmk = 0$.

Finally, let $k_2 = m_1 + \cdots + m_\ell$ be an arbitrary element of K, then it follows from the previous paragraph that $vk_2k = vm_1k + \cdots + vm_\ell k = 0$, which completes the proof of this lemma. □

Lemma 2.7.22. *For all $v_1, v_2 \in V$ and all $k \in K$, we have that*

$$F(v_1 k, v_2) = F(v_1, v_2 k^\sigma) \ .$$

Proof. Since both V and W are abelian, it follows from $(\mathbf{Q}_{21})$ and $(\mathbf{Q}_{11})$ that $F(v_1 w, v_2) = F(v_1, v_2 w)$ for all $v_1, v_2 \in V$ and all $w \in W$. For $m \in M$, say $m = w_1 \bullet \cdots \bullet w_n$, it follows by induction on n that $F(v_1 m, v_2) = F(v_1, v_2 m^\sigma)$. Since F is additive in both variables, it now follows that $F(v_1 k, v_2) = F(v_1, v_2 k^\sigma)$ for any $k = m_1 + \cdots + m_\ell \in K$. □

Our first goal is to prove that Δ is a quadrangular system of involutory type. Assume from now on that $V = E$, that is, that $\Omega = \Delta$.

Lemma 2.7.23. (i) *Let $k \in K$. If $\epsilon k = 0$, then $k = 0$.*

(ii) *Let $k_1, k_2 \in K$. If $\epsilon k_1 = \epsilon k_2$, then $k_1 = k_2$.*

Proof. Let $k \in K$ be such that $\epsilon k = 0$. By Lemma 2.7.21, it follows that $\epsilon k_2 k = 0$ for all $k_2 \in K$, and hence $Vk = Ek = \epsilon K k = 0$, which implies that $k = 0$ (remember that the elements of K are endomorphisms of V). This proves (i); (ii) now follows from (i) by substituting $k_1 - k_2$ for k. □

Definition 2.7.24. For each $k \in K^*$, we define k' as the (unique!) element in K such that $(\epsilon k)^{-1} = \epsilon k'$.

Lemma 2.7.25. *For all $k \in K^*$ and all $w \in W$, we have that*

$$k \bullet w = w(\epsilon k) \bullet (k')^\sigma \ .$$

Proof. By $(\mathbf{Q}_{16})$, we have that

$$\begin{aligned}
(\epsilon k) w &= \overline{(\epsilon k)^{-1} \cdot w(\epsilon k)} \\
&= \overline{\epsilon \cdot k' \cdot w(\epsilon k)} \\
&= \overline{\epsilon \cdot (k' \bullet w(\epsilon k))} \\
&= \epsilon \cdot (w(\epsilon k) \bullet (k')^\sigma) \ ,
\end{aligned}$$

where we have used Theorem 2.7.19. It follows by Lemma 2.7.23(ii) that $k \bullet w = w(\epsilon k) \bullet (k')^\sigma$. □

Lemma 2.7.26. *For all $k \in K^*$ and all $w \in W^*$, we have that*

$$(k \bullet w)' = w^{-1} \bullet k' \ .$$

Proof. By Lemma 2.7.25 with $k \bullet w$ in place of k and w^{-1} in place of w and by ($\mathbf{Q}_{19}$), we have that

$$\begin{aligned} k \bullet w &= ((k \bullet w) \bullet w^{-1}) \bullet w \\ &= w^{-1}(\epsilon(k \bullet w)) \bullet (k \bullet w)'^{\sigma} \bullet w \\ &= w(\epsilon k) \bullet (k \bullet w)'^{\sigma} \bullet w \,, \end{aligned}$$

which, together with Lemma 2.7.25, implies that

$$w(\epsilon k) \bullet (k')^{\sigma} \bullet w^{-1} = w(\epsilon k) \bullet (k \bullet w)'^{\sigma} \,.$$

If we apply σ to both sides, we get that

$$w^{-1} \bullet k' \bullet w(\epsilon k) = (k \bullet w)' \bullet w(\epsilon k) \,,$$

from which it follows that $w^{-1} \bullet k' = (k \bullet w)'$ since $w(\epsilon k)$ is invertible in K. □

Lemma 2.7.27. *For all $w_1, \ldots, w_n \in W^*$, we have that*

$$(w_1 \bullet \cdots \bullet w_n)' = w_n^{-1} \bullet \cdots \bullet w_1^{-1} \,.$$

Proof. By Lemma 2.7.26 with $k = \delta$, we have that $w_1' = w_1^{-1}$. Again by Lemma 2.7.26, it now follows by induction on n that

$$\begin{aligned} (w_1 \bullet \cdots \bullet w_n)' &= w_n^{-1} \bullet (w_1 \bullet \cdots \bullet w_{n-1})' \\ &= w_n^{-1} \bullet w_{n-1}^{-1} \bullet \cdots \bullet w_1^{-1} \,, \end{aligned}$$

which is what we wanted to show. □

Since it follows from this lemma that $m \bullet m' = m' \bullet m = \delta$, we will from now on write m^{-1} in place of m' for all $m \in M$.

Lemma 2.7.28. *For all $m \in M$ and all $w \in W$, we have that*

$$w(\epsilon m) = m \bullet w \bullet m^{\sigma} \,.$$

Proof. First of all, observe that it follows from Lemma 2.7.27 that $(m^{-1})^{\sigma} = (m^{\sigma})^{-1}$. By Lemma 2.7.25, we have that $m \bullet w = w(\epsilon m) \bullet (m^{-1})^{\sigma}$. It follows that $m \bullet w \bullet m^{\sigma} = w(\epsilon m)$. □

Lemma 2.7.29. *For all $k_1, k_2 \in K$, we have that*

$$F(\epsilon k_1, \epsilon k_2) = k_1 \bullet k_2^{\sigma} + k_2 \bullet k_1^{\sigma} \,.$$

Proof. Since $v+\overline{v} = \epsilon F(\epsilon, v)$ for all $v \in V$, we have that $\epsilon F(\epsilon, \epsilon k) = \epsilon k + \overline{\epsilon k} = \epsilon k + \epsilon k^\sigma$ and hence $F(\epsilon, \epsilon k) = k + k^\sigma$, for all $k \in K$. It now follows from Lemma 2.7.22 that

$$\begin{aligned} F(\epsilon k_1, \epsilon k_2) &= F(\epsilon, \epsilon k_2 k_1^\sigma) \\ &= F(\epsilon, \epsilon(k_2 \bullet k_1^\sigma)) \\ &= k_2 \bullet k_1^\sigma + k_1 \bullet k_2^\sigma \ , \end{aligned}$$

which proves the lemma. □

Theorem 2.7.30. *For all $k \in K$ and all $w \in W$, we have that*

$$w(\epsilon k) = k \bullet w \bullet k^\sigma \ .$$

Proof. In Lemma 2.7.28, we have shown this theorem for all $k \in M$. Now suppose that the theorem holds for $k_1, k_2 \in K$. We will show that it then holds for $k_1 + k_2$ as well, which will prove the theorem for all $k \in K$.

It follows from $(\mathbf{Q}_{11})$ and Lemma 2.7.29 that

$$\begin{aligned} w(\epsilon(k_1+k_2)) &= w(\epsilon k_1 + \epsilon k_2) \\ &= w(\epsilon k_1) + w(\epsilon k_2) + F(\epsilon k_2 w, \epsilon k_1) \\ &= k_1 \bullet w \bullet k_1^\sigma + k_2 \bullet w \bullet k_2^\sigma + (k_2 \bullet w) \bullet k_1^\sigma + k_1 \bullet (k_2 \bullet w)^\sigma \\ &= k_1 \bullet w \bullet k_1^\sigma + k_2 \bullet w \bullet k_2^\sigma + k_2 \bullet w \bullet k_1^\sigma + k_1 \bullet w \bullet k_2^\sigma \\ &= (k_1 + k_2) \bullet w \bullet (k_1+k_2)^\sigma \ , \end{aligned}$$

which is what we had to show. □

Theorem 2.7.31. *$K_{+,\bullet}$ is a field or a skew-field.*

Proof. We already know that $K_{+,\bullet}$ is a ring. Let k be an arbitrary element of K^*. We will show that $k' \bullet k = k \bullet k' = \delta$.

By $(\mathbf{Q}_{15})$, Theorem 2.7.30 and Lemma 2.7.25, we have that

$$\begin{aligned} \delta &= \delta(\epsilon k)(\epsilon k)^{-1} \\ &= \delta(\epsilon k)(\epsilon k') \\ &= k' \bullet \delta(\epsilon k) \bullet (k')^\sigma \\ &= k' \bullet (k \bullet \delta) \\ &= k' \bullet k \ , \end{aligned}$$

and if we substitute k' for k, then we get that $\delta = k \bullet k'$ as well, since it follows from the definition of k' that $k'' = k$.

Hence every non-zero element $k \in K^*$ is invertible with inverse $k^{-1} = k'$. It follows that $K_{+,\bullet}$ is a field or a skew-field. □

For technical reasons which will be clear in a moment, we now define $K_{+,\cdot} := K^{\mathrm{op}}_{+,\bullet}$, that is, we set $k_1k_2 := k_2 \bullet k_1$ for all $k_1, k_2 \in K$.

Theorem 2.7.32. *(K, W, σ) is an involutory set. Furthermore, K is generated by W as a ring.*

Proof. We have just shown that K is a field or a skew-field. It is obvious from the definition that $\sigma^2 = 1$ and that $(k_1k_2)^\sigma = (k_2 \bullet k_1)^\sigma = k_1^\sigma \bullet k_2^\sigma = k_2^\sigma k_1^\sigma$ for all $k_1, k_2 \in K$, so σ is an involution of K. W is an additive subgroup of K containing δ. By Lemma 2.7.29, $k + k^\sigma = F(\epsilon, \epsilon k) \in \mathrm{Im}(F) \subseteq W$ for all $k \in K$, hence $K_\sigma \subseteq W$, and by the definition of σ, all elements of W are fixed by σ. Finally, it follows from Theorem 2.7.30 that $k^\sigma W k = k \bullet W \bullet k^\sigma \subseteq W(\epsilon k) \subseteq W$ for all $k \in K$. Thus (K, W, σ) is an involutory set.

The fact that K is generated by W as a ring follows immediately from the definition of the ring K. □

Theorem 2.7.33. $(E, W, \tau_V, \tau_W, \epsilon, \delta) \cong \Omega_I(K, W, \sigma)$.

Proof. Let ϕ be the isomorphism from $[K]$ to E which maps $[k]$ to ϵk for all $k \in K$, and let ψ be the isomorphism from $[W]$ to W which maps $[w]$ to w for all $w \in W$. Then $\phi([\delta]) = \epsilon\delta = \epsilon$ and $\psi([\delta]) = \delta$. Furthermore,

$$\phi([k][w]) = \phi([wk]) = \epsilon(wk) = \epsilon(k \bullet w) = \epsilon k \cdot w = \phi([k])\psi([w])\ , \text{ and}$$
$$\psi([w][k]) = \psi(k^\sigma wk) = k^\sigma wk = k \bullet w \bullet k^\sigma = w(\epsilon k) = \psi([w])\phi([k])\ ,$$

for all $w \in W$ and all $k \in K$. Hence (ϕ, ψ) is an isomorphism from $\Omega_I(K, W, \sigma)$ to $(E, W, \tau_V, \tau_W, \epsilon, \delta)$. □

The next lemma shows the "$\sigma \neq 1$" part of Theorem 2.7.6.

Lemma 2.7.34. *If (K, K_0, σ) is an involutory set with $\sigma = 1$, then $\Omega_I(K, K_0, \sigma)$ is normal or indifferent.*

Proof. Since σ is an involution, $ab = (ab)^\sigma = b^\sigma a^\sigma = ba$ for all $a, b \in K$, hence K is abelian. It follows that $F([a], [b]) = [2ab]$ for all $a, b \in K$. If $\mathrm{char}(K) = 2$, then $F \equiv 0$, hence $\Omega_I(K, K_0, \sigma)$ is indifferent. So we can assume that $\mathrm{char}(K) \neq 2$. But then $K_\sigma = \{2a \mid a \in K\} = K$, and hence $K_0 = K$. It follows that for all elements $t_1, t_2, \ldots, t_n \in K_0$, the product $t_1t_2 \ldots t_n$ lies in K_0 as well, and hence $[1][t_1][t_2] \ldots [t_n] = [1][t_1t_2 \ldots t_n]$, which implies that $\Omega_I(K, K_0, \sigma)$ is normal. □

From now on, we drop our assumption that $V = E$ (but we still assume that $H \equiv 0$). Our next goal is to show that if Ω is reduced but not normal, then $V = E$ after all.

We start with a generalization of Lemma 2.7.18:

Lemma 2.7.35. *For all $v \in V^*$, all $c \in V$ and all $k_1, k_2 \in K$, we have that*

$$\pi_v(\pi_v(c)k_1)k_2 = \pi_v(\pi_v(ck_2)k_1) \ .$$

Proof. If we substitute $\overline{c}(\delta v)^{-1}$ for c in ($\mathbf{Q}_{25}$), then we get that $\pi_v(c)w = \pi_v\big(\overline{\overline{c}(\delta v)^{-1}(wv)}\big)$, and hence $\pi_v(\pi_v(c)w) = \overline{\overline{c}(\delta v)^{-1}(wv)}$ for all $c, v \in V$ and all $w \in W$. It thus follows by 2.2.23(i) and Lemma 2.7.18 that

$$\begin{aligned} \pi_v(\pi_v(c)w_1)w_2 &= \overline{\overline{c}(\delta v)^{-1}(w_1 v)}w_2 \\ &= \overline{\overline{cw_2}(\delta v)^{-1}(w_1 v)} \\ &= \pi_v(\pi_v(cw_2)w_1) \ , \end{aligned}$$

which shows the lemma for all $k_1, k_2 \in W$. In the same way as in Lemma 2.7.17 and Lemma 2.7.18, we can use induction to deduce from this that the lemma holds for all $k_1, k_2 \in M$. Since π_v is additive, it then follows that the lemma holds for all $k_1, k_2 \in K$. □

Lemma 2.7.36. *For all $v \in V^*$ and all $k \in K$, we have that $\pi_v(vk) = -vk^\sigma$.*

Proof. Since π_v is additive, it suffices to show that $\pi_v(vm) = -vm^\sigma$ for all $m = w_1 \bullet \cdots \bullet w_n \in M$, which we will do by induction on n.

It already follows from 2.2.21 that $\pi_v(vw_1) = -vw_1$, which shows the statement for $n = 1$. Now assume that $\pi_v(vw_1 \dots w_{n-1}) = -vw_{n-1} \dots w_1$ for all $w_1, \dots, w_{n-1} \in W$. Then it follows by Lemma 2.7.35 that

$$\begin{aligned} \pi_v(vw_1 \dots w_n) &= -\pi_v(\pi_v(vw_{n-1} \dots w_1)w_n) \\ &= -\pi_v(\pi_v(v)w_n)w_{n-1} \dots w_1 \\ &= -vw_n w_{n-1} \dots w_1 \ , \end{aligned}$$

since $\pi_v(\pi_v(v)w_n) = -\pi_v(vw_n) = vw_n$. □

Lemma 2.7.37. *For all $v \in V$ and all $w \in W$, we have that $vww = v \cdot \delta(\epsilon w)$.*

Proof. We may assume that $v \neq 0$. Since $H \equiv 0$, it follows from ($\mathbf{Q}_{26}$) that $\delta(\epsilon w)v = \delta(vw)$, and hence $\delta(\epsilon w) = \delta(vw)v^{-1}$. By ($\mathbf{Q}_{16}$), it follows that

$$v \cdot \delta(\epsilon w) = v \cdot \big(\delta(vw)v^{-1}\big) = \overline{v^{-1} \cdot \delta(vw)} \ .$$

If we substitute vw for v, v^{-1} for c and w^{-1} for w in ($\mathbf{Q}_{25}$), then we get, by ($\mathbf{Q}_{19}$), ($\mathbf{Q}_{16}$) and 2.2.21, that

$$\begin{aligned} \pi_{vw}(\overline{v^{-1} \cdot \delta(vw)})w^{-1} &= \pi_{vw}(\overline{v^{-1} \cdot w^{-1}(vw)}) \\ &= \pi_{vw}(\overline{v^{-1} \cdot wv}) \\ &= \pi_{vw}(vw) \\ &= -vw \ , \end{aligned}$$

and hence $\pi_{vw}(v \cdot \delta(\epsilon w)) = -vww$, from which it follows, by 2.2.23(i), that $v \cdot \delta(\epsilon w) = -\pi_{vw}(vww) = vww$ by Lemma 2.7.36, which is what we had to show. $\square$

Lemma 2.7.38. *For all $v \in V$ and all $w, z \in W$, we have that*

$$vwzw = v \cdot z(\epsilon w) .$$

Proof. We may assume that $v \neq 0$. Since $H \equiv 0$, it follows from $(\mathbf{Q}_{26})$ that $z(\epsilon w)v = z(vw)$. By $(\mathbf{Q}_{16})$, it follows that

$$v \cdot z(\epsilon w) = v \cdot \big(z(vw)v^{-1}\big) = \overline{v^{-1} \cdot z(vw)} .$$

If we substitute vw for v, v^{-1} for c and z for w in $(\mathbf{Q}_{25})$, then we get that

$$\pi_{vw}(\overline{v^{-1} \cdot \delta(vw)})z = \pi_{vw}(\overline{v^{-1} \cdot z(vw)}) ,$$

and hence

$$\pi_{vw}(v \cdot \delta(\epsilon w))z = \pi_{vw}(v \cdot z(\epsilon w)) .$$

It now follows from Lemma 2.7.37 and Lemma 2.7.36 that

$$\begin{aligned} v \cdot z(\epsilon w) &= \pi_{vw}(\pi_{vw}(vww)z) \\ &= \pi_{vw}(-vwwz) \\ &= vwzw , \end{aligned}$$

which proves the lemma. $\square$

Lemma 2.7.39. *For all $v \in V$, all $z \in W$ and all $m \in M$, we have that*

$$vmzm^\sigma = v \cdot z(\epsilon m) .$$

In particular, $vmm^\sigma = v \cdot \delta(\epsilon m)$.

Proof. Let $m = w_1 \bullet \cdots \bullet w_n$ with $w_1, \dots, w_n \in W^*$. We will prove the lemma by induction on n.

We have already shown in Lemma 2.7.38 that the current lemma holds for $n = 1$. Now assume that

$$vw_1 \dots w_{n-1} y w_{n-1} \dots w_1 = v \cdot y(\epsilon w_1 \dots w_{n-1})$$

for all $y \in W$. Then by Lemma 2.7.38 and $(\mathbf{Q}_{26})$, we have that

$$\begin{aligned} vw_1 \dots w_n z w_n \dots w_1 &= \big((vw_1 \dots w_{n-1})w_n z w_n\big) w_{n-1} \dots w_1 \\ &= vw_1 \dots w_{n-1} \cdot z(\epsilon w_n) \cdot w_{n-1} \dots w_1 \\ &= v \cdot z(\epsilon w_n)(\epsilon w_1 \dots w_{n-1}) \\ &= v \cdot z(\epsilon w_1 \dots w_{n-1} w_n) , \end{aligned}$$

and we are done. $\square$

Lemma 2.7.40. *For all $v \in V$ and all $w_1, w_2, w_3 \in W$, we have that*

(i) $F(vw_1, vw_2w_3) = F(\epsilon w_1, \epsilon w_2 w_3)v$;

(ii) $vw_1w_2w_3 + vw_3w_2w_1 = vF(\epsilon w_1, \epsilon w_3 w_2)$.

Proof. By ($\mathbf{Q}_{21}$), ($\mathbf{Q}_{11}$) and ($\mathbf{Q}_{26}$), we have that

$$\begin{aligned} F(vw_1, vw_2w_3) &= w_3(vw_1 + vw_2) - w_3(vw_1) - w_3(vw_2) \\ &= w_3 \cdot v(w_1 + w_2) - w_3 \cdot vw_1 - w_3 \cdot vw_2 \\ &= w_3 \cdot \epsilon(w_1 + w_2) \cdot v - w_3 \cdot \epsilon w_1 \cdot v - w_3 \cdot \epsilon w_2 \cdot v \\ &= (w_3 \cdot (\epsilon w_1 + \epsilon w_2) - w_3 \cdot \epsilon w_1 - w_3 \cdot \epsilon w_2) \cdot v \\ &= F(\epsilon w_1, \epsilon w_2 w_3)v \ , \end{aligned}$$

which proves (i). By Lemma 2.7.36, the definition of π_v, Lemma 2.7.22, (i) and ($\mathbf{Q}_{16}$), we have that

$$\begin{aligned} vw_1w_2w_3 + vw_3w_2w_1 &= vw_3w_2w_1 - \pi_v(vw_3w_2w_1) \\ &= \overline{v^{-1}F(v, vw_3w_2w_1)} \\ &= \overline{v^{-1}F(vw_1, vw_3w_2)} \\ &= \overline{v^{-1} \cdot F(\epsilon w_1, \epsilon w_3 w_2)v} \\ &= vF(\epsilon w_1, \epsilon w_3 w_2) \ , \end{aligned}$$

which proves (ii). □

Lemma 2.7.41. *Let $w_1, w_2, w_3 \in W$ be arbitrary. Let $k = w_1 + w_2 \bullet w_3 \in K$. Then*

(i) $vkk^\sigma = v \cdot \delta(\epsilon k)$ *for all* $v \in V$;

(ii) *If* $\epsilon k = 0$, *then* $k = 0$.

Proof. By Lemma 2.7.39, Lemma 2.7.40(ii) and ($\mathbf{Q}_{11}$), we have that

$$\begin{aligned} vkk^\sigma &= v(w_1 + w_2 \bullet w_3)(w_1 + w_3 \bullet w_2) \\ &= vw_1w_1 + vw_2w_3w_3w_2 + vw_1w_3w_2 + vw_2w_3w_1 \\ &= v \cdot \delta(\epsilon w_1) + v \cdot \delta(\epsilon w_2 w_3) + vF(\epsilon w_1, \epsilon w_2 w_3) \\ &= v \cdot (\delta(\epsilon w_1) + \delta(\epsilon w_2 w_3) + F(\epsilon w_1, \epsilon w_2 w_3)) \\ &= v \cdot \delta(\epsilon w_1 + \epsilon w_2 w_3) \\ &= v \cdot \delta(\epsilon k) \end{aligned}$$

for all $v \in V$, which proves (i).

Now suppose that $\epsilon k = \epsilon w_1 + \epsilon w_2 w_3 = 0$. Then it follows, by $(\mathbf{Q}_{11})$, Lemma 2.7.15 and Lemma 2.7.40(i), that

$$\begin{aligned}
0 &= \delta \cdot (\epsilon w_1 + \epsilon w_2 w_3) \cdot v \\
&= \delta \cdot \epsilon w_1 \cdot v + \delta \cdot \epsilon w_2 w_3 \cdot v + F(\epsilon w_1, \epsilon w_2 w_3) \cdot v \\
&= \delta \cdot v w_1 + \delta \cdot v w_2 w_3 + F(v w_1, v w_2 w_3) \\
&= \delta \cdot (v w_1 + v w_2 w_3) \\
&= \delta \cdot vk \,,
\end{aligned}$$

and hence $vk = 0$, for all $v \in V$. So $k = 0$. $\square$

Remark 2.7.42. It will follow from the classification that the statements in Lemma 2.7.40 and Lemma 2.7.41 actually hold in a much broader generality, for all reduced quadrangular systems. More precisely, we have that

(i) $F(vk_1, vk_2) = F(\epsilon k_1, \epsilon k_2)v$;

(ii) $vk + vk^\sigma = vF(\epsilon k, \epsilon)$;

(iii) $vkk^\sigma = v \cdot \delta(\epsilon k)$;

(iv) If $\epsilon k = 0$, then $k = 0$;

for all $v \in V$ and all $k, k_1, k_2 \in K$. However, we are not aware of a simple proof of these facts at this step of the classification.

By definition, $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ is normal if and only if $\epsilon M = \epsilon W$. To complete the proof of Theorem 2.7.6, it will thus suffice to prove the following theorem:

Theorem 2.7.43. *If $V \neq E$ and $F \not\equiv 0$, then $\epsilon M = \epsilon W$. Furthermore, $vm = vm^\sigma$ and $\overline{vm} = \overline{v}m$ for all $v \in V \setminus E$ and all $m \in M$.*

Proof. We start by showing that $\overline{vw} = \overline{v}w$ for all $v \in V \setminus E$ and all $w \in W$. So let $v \in V \setminus E$ and $w \in W$ be arbitrary. By 2.2.20, $\overline{\overline{v}w} = v \cdot \delta v^{-1} \cdot w\overline{v}$, and hence $vw - \overline{\overline{v}w} = vk$ for $k = w - \delta v^{-1} \bullet w\overline{v}$. On the other hand,

$$\begin{aligned}
vw - \overline{\overline{v}w} &= vw - \overline{\epsilon F(v, \epsilon)w - vw} \\
&= vw - \epsilon F(\epsilon F(v, \epsilon)w - vw, \epsilon) + \epsilon F(v, \epsilon)w - vw \\
&= \epsilon F(v, \epsilon)w - \epsilon F(\epsilon F(v, \epsilon)w - vw, \epsilon) \\
&\in \epsilon K \,,
\end{aligned}$$

hence $vk \in \epsilon K$. Suppose that $vk \neq 0$. Then it would follow from Lemma 2.7.41 that $\epsilon k \neq 0$ and that $vkk^\sigma \cdot (\delta(\epsilon k))^{-1} = v$. Hence we would have $v \in \epsilon K \cdot k^\sigma \cdot (\delta(\epsilon k))^{-1} \subseteq \epsilon K = E$, which contradicts the choice of v. So we must have $vk = 0$, and thus $\overline{\overline{v}w} = vw$, which shows that $\overline{vw} = \overline{v}w$.

We will now show by induction on n that $\overline{vw_1 \dots w_n} = \overline{v}w_1 \dots w_n$ for all $v \in V \setminus E$ and all $w_1, \dots, w_n \in W$. We have already shown this for $n = 1$, so suppose that it holds for $n-1$. We may assume that $w_i \neq 0$ for all $i \in \{1, \dots, n\}$. If $v \in V \setminus E$, then also $vw_1 \dots w_{n-1} \in V \setminus E$, since $v = vw_1 \dots w_{n-1}w_{n-1}^{-1} \dots w_1^{-1}$. Hence we can substitute $vw_1 \dots w_{n-1}$ for v in the result of the previous paragraph, and we get that

$$\begin{aligned} \overline{vw_1 \dots w_n} &= \overline{vw_1 \dots w_{n-1} \cdot w_n} \\ &= \overline{vw_1 \dots w_{n-1}} \cdot w_n \\ &= \overline{v}w_1 \dots w_n \ ; \end{aligned}$$

the statement thus holds for n as well. This shows that $\overline{vm} = \overline{v}m$ for all $v \in V \setminus E$ and all $m \in M$.

We will now prove by induction on n that $\overline{vw_1 \dots w_n} = \overline{v}w_n \dots w_1$ for all $v \in V$ and all $w_1, \dots, w_n \in W$. Again, we have already shown this for $n = 1$, so suppose that it holds for $n-1$. Then, by Lemma 2.7.17,

$$\begin{aligned} \overline{vw_1 \dots w_n} &= \overline{\overline{\overline{v}w_{n-1} \dots w_1}w_n} \\ &= \overline{\overline{v}w_n}w_{n-1} \dots w_1 \\ &= \overline{v}w_n \dots w_1 \ , \end{aligned}$$

so it holds for n as well. We have thus proved that $\overline{vm} = \overline{v}m^\sigma$ for all $m \in M$.

Since $v \in V \setminus E$ if and only if $\overline{v} \in V \setminus E$, it follows from the previous two paragraphs that $vm = vm^\sigma$ for all $v \in V \setminus E$ and all $m \in M$.

We first assume that $\epsilon \notin \mathrm{Rad}(F)$. Since E is a proper subgroup of V, V is generated by $V \setminus E$. Since $\epsilon \notin \mathrm{Rad}(F)$, this implies that $F(\epsilon, V \setminus E) \neq 0$, so there exists an element $v \in V \setminus E$ such that $F(\epsilon, v) \neq 0$. Let m be an arbitrary element of M, and let $m_2 := F(\epsilon, v)^{-1} \bullet m \in M$. Then it follows from $\overline{vm_2} = \overline{v}m_2$ that

$$\epsilon F(\epsilon, vm_2) - vm_2 = \epsilon F(\epsilon, v)m_2 - vm_2$$

and hence

$$\epsilon F(\epsilon, vF(\epsilon, v)^{-1}m) = \epsilon F(\epsilon, v)F(\epsilon, v)^{-1}m$$

from which it follows that

$$\epsilon m = \epsilon F(\epsilon, vF(\epsilon, v)^{-1}m) \in \epsilon W \ ,$$

for all $m \in M$. So we have shown that $\epsilon M = \epsilon W$ in this case.

Now assume that $\epsilon \in \mathrm{Rad}(F)$. By Lemma 2.7.11, all elements of V and W have order at most 2, and $\overline{v} = v$ for all $v \in V$.

Since $F \not\equiv 0$, there exists an element $\eta \in V \setminus \mathrm{Rad}(F)$. We have that $\mathrm{Rad}(F) \cdot K = \mathrm{Rad}(F)$ by Lemma 2.7.22, so $E = \epsilon K \subseteq \mathrm{Rad}(F)$, and hence $\eta \in V \setminus E$. It follows that $\eta m = \eta m^\sigma$ for all $m \in M$, which implies, by Lemma 2.7.36, that $\pi_\eta(\eta m) = \eta m^\sigma = \eta m$. By the definition of π_η, this implies that $\eta^{-1}F(\eta, \eta m) = 0$, and hence $F(\eta, \eta m) = 0$ for all $m \in M$. Since F is additive, this in turn implies that $F(\eta, \eta K) = 0$. Since we chose $\eta \notin \mathrm{Rad}(F)$, we conclude that $V \neq \eta K$.

We now show that $\pi_\eta(vw) = \pi_\eta(v)w$ for all $v \in V \setminus \eta K$ and all $w \in W$. So let $v \in V \setminus \eta K$ and $w \in W$ be arbitrary. If we substitute η for v and $v(\delta\eta)^{-1}$ for c in ($\mathbf{Q}_{25}$), then we get that $\pi_\eta(v)w = \pi_\eta(v(\delta\eta)^{-1}(w\eta))$, and hence $\pi_\eta(\pi_\eta(v)w) = v(\delta\eta)^{-1}(w\eta)$. Hence $vw + \pi_\eta(\pi_\eta(v)w) = vk$ where $k = w + (\delta\eta)^{-1} \bullet (w\eta) \in K$. On the other hand,

$$\begin{aligned} vw + \pi_\eta(\pi_\eta(v)w) &= vw + \pi_\eta(vw + \eta F(\eta^{-1}, v)w) \\ &= vw + vw + \eta F(\eta^{-1}, v)w + \eta F(\eta^{-1}, vw + \eta F(\eta^{-1}, v)w) \\ &= \eta F(\eta^{-1}, v)w + \eta F(\eta^{-1}, vw + \eta F(\eta^{-1}, v)w) \\ &\in \eta K, \end{aligned}$$

hence $vk \in \eta K$. In a similar way as in the first paragraph, it would follow from $vk \neq 0$ that $v \in \eta K$, which would contradict the choice of v. Hence $vk = vw + \pi_\eta(\pi_\eta(v)w) = 0$, and thus $\pi_\eta(vw) = \pi_\eta(v)w$.

Again, it follows by induction on n that $\pi_\eta(vw_1 \dots w_n) = \pi_\eta(v)w_1 \dots w_n$ for all $v \in V \setminus \eta K$ and all $w_1, \dots w_n \in W$, that is, $\pi_\eta(vm) = \pi_\eta(v)m$ for all $v \in V \setminus \eta K$ and all $m \in M$.

Since ηK is a proper subgroup of V, V is generated by $V \setminus \eta K$. Since $\eta \notin \mathrm{Rad}(F)$, this implies that $F(\eta, V \setminus \eta K) \neq 0$, so there exists an element $v \in V \setminus \eta K$ such that $F(\eta, v) \neq 0$. It follows from ($\mathbf{Q}_{17}$) that $F(\eta^{-1}, v) = F(\eta, v)\eta^{-1} \neq 0$ as well.

Let m be an arbitrary element of M, and let $m_2 := F(\eta^{-1}, v)^{-1} \bullet m \in M$. Then it follows from $\pi_\eta(vm_2) = \pi_\eta(v)m_2$ that

$$vm_2 + \eta F(\eta^{-1}, vm_2) = vm_2 + \eta F(\eta^{-1}, v)m_2$$

and hence

$$\eta F(\eta^{-1}, vF(\eta^{-1}, v)^{-1}m) = \eta F(\eta^{-1}, v)F(\eta^{-1}, v)^{-1}m \ ,$$

from which it follows that

$$\eta m = \eta F(\eta^{-1}, vF(\eta^{-1}, v)^{-1}m) \ ,$$

for all $m \in M$. So we have shown that $\eta M = \eta W$.

Since $\eta \in V \setminus \mathrm{Rad}(F)$ and $\epsilon \in \mathrm{Rad}(F)$, we have $\eta + \epsilon \in V \setminus \mathrm{Rad}(F)$ as well. So the conclusion of the previous paragraph is also valid for $\eta + \epsilon$, that is, $(\eta + \epsilon)M = (\eta + \epsilon)W$. Now let m be an arbitrary element of M. Then $\eta m = \eta w_1$ and $(\eta + \epsilon)m = (\eta + \epsilon)w_2$ for some $w_1, w_2 \in W$. It follows that

$$\begin{aligned} \epsilon m &= \eta m + (\eta + \epsilon)m \\ &= \eta w_1 + (\eta + \epsilon)w_2 \\ &= \eta(w_1 + w_2) + \epsilon w_2 \, . \end{aligned}$$

If $w_1 + w_2 \neq 0$, then it would follow from this that

$$\eta = (\epsilon m + \epsilon w_2) \cdot (w_1 + w_2)^{-1} \in \epsilon K = E \subseteq \mathrm{Rad}(F) \, ,$$

which contradicts the choice of η. Hence we must have $w_1 + w_2 = 0$, and it follows that $\epsilon m = \epsilon w_2$. Since m was arbitrary, we have shown that $\epsilon M = \epsilon K$ also in this case.

This completes the proof of this theorem, and thereby also the proof of Theorem 2.7.6. □

2.7.2 Quadrangular Systems of Quadratic Form Type

Our goal in this section is to classify the quadrangular systems which are normal.

Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system which is normal. In particular, Ω is reduced, so $H \equiv 0$.

Lemma 2.7.44. *K is abelian, i.e. $vk_1k_2 = vk_2k_1$ for all $v \in V$ and all $k_1, k_2 \in K$. Equivalently, $\sigma = 1$.*

Proof. Note that by the definition of σ, K is abelian if and only if $\sigma = 1$.

It suffices to show that $vw_1w_2 = vw_2w_1$ for all $v \in V$ and all $w_1, w_2 \in W$. If $v \in V \setminus E$, then this follows by substituting $w_1 \bullet w_2$ for m in Theorem 2.7.43. If $v = \epsilon$, then $\epsilon w_1 w_2 = \epsilon w_3$ for some $w_3 \in W$ since Ω is normal. Hence, by Theorem 2.7.19, $\epsilon w_1 w_2 = \epsilon w_3 = \overline{\epsilon w_3} = \overline{\epsilon w_1 w_2} = \epsilon w_2 w_1$. Finally, assume that $v = \epsilon k$ for some $k \in K$. Then $vw_1w_2 = \epsilon k w_1 w_2 = \overline{\epsilon w_2 w_1 k^\sigma} = \overline{\epsilon w_1 w_2 k^\sigma} = \epsilon k w_2 w_1 = v w_2 w_1$, again by Theorem 2.7.19. This shows the lemma in all possible cases. □

Lemma 2.7.45. *For all $v \in V$, we have that $vK = vW$.*

Proof. It suffices to show that $vM = vW$.

First, assume that $v \in E$, say $v = \epsilon k$ for some $k \in K$. Let $m \in M$ be arbitrary. Then $\epsilon m = \epsilon w$ for some $w \in W$, since Ω is normal. Since K is abelian by Lemma 2.7.44, it follows that $vm = \epsilon km = \epsilon mk = \epsilon wk = \epsilon kw = vw$, which shows that $vM = vW$ in this case.

Now, assume that $v \in V \setminus E$. Note that it is sufficient to show that $vw_1w_2 \in vW$ for all $w_1, w_2 \in W$; it then follows by induction that $vM = vW$. Choose two arbitrary elements $w_1, w_2 \in W$. By Lemma 2.7.40(ii) and Lemma 2.7.44, we have that $vw_1w_2w_3 + vw_1w_2w_3 = vF(\epsilon w_1, \epsilon w_2w_3)$, or equivalently, $vw_1w_2w_3(\delta + \delta) = vF(\epsilon w_1, \epsilon w_2w_3)$, for all $w_3 \in W$.

We now distinguish two cases. First, assume that $\delta + \delta = 0$. It then follows, by ($\mathbf{Q}_{12}$), that all elements of V and W have order at most 2. Since Ω is normal, there exists a $w \in W$ such that $\epsilon w_1w_2 = \epsilon w$. By the previous paragraph, $F(\epsilon w_1, \epsilon w_2w) = 0$, and hence, by Lemma 2.7.22 and Lemma 2.7.40(i), $F(vw_1w_2, vw) = F(vw_1, vw_2w) = F(\epsilon w_1, \epsilon w_2w)v = 0$ as well. By Lemma 2.7.15, it follows that

$$\begin{aligned}\delta(vw_1w_2 + vw) &= \delta \cdot vw_1w_2 + \delta \cdot vw + F(vw_1w_2, vw) \\ &= \delta \cdot \epsilon w_1w_2 \cdot v + \delta \cdot \epsilon w \cdot v \\ &= \delta \cdot \epsilon w \cdot v + \delta \cdot \epsilon w \cdot v \\ &= 0\,,\end{aligned}$$

which implies that $vw_1w_2 + vw = 0$, hence $vw_1w_2 = vw \in vW$, which is what we had to show.

Now, assume that $\delta + \delta \neq 0$. Then we set $w_3 = (\delta + \delta)^{-1}$ in the identity $vw_1w_2w_3(\delta + \delta) = vF(\epsilon w_1, \epsilon w_2w_3)$, which yields

$$vw_1w_2 = vF(\epsilon w_1, \epsilon w_2(\delta + \delta)^{-1}) \in vW\,,$$

which proves the lemma in this case as well. □

Lemma 2.7.46. *Let $w \in W$ and $k \in K$ be such that $vk = vw$ for some $v \in V^*$. Then $k = w$.*

Proof. We will show that $uk = uw$ for all $u \in V$. We distinguish two cases.

First, assume that $u \in vK$. Since $vK = vW$ by Lemma 2.7.45, there exists a $z \in W$ such that $u = vz$. Then $uk = vzk = vkz = vwz = vzw = uw$, since K is abelian.

Now, assume that $u \notin vK$. By Lemma 2.7.45, there exists a $w_2 \in W$ such that $uk = uw_2$, and there exists a $w_3 \in W$ such that $(v + u)k = (v + u)w_3$. We have to show that $w = w_2$. We have that $vw_3 + uw_3 = (v + u)w_3 = (v + u)k = vk + uk = vw + uw_2$, from which it follows that

$u(w_3 - w_2) = v(w - w_3)$. Since $u \notin vK$, this can only occur if $w_3 - w_2 = 0$, and then $w - w_3 = 0$ as well. Hence $w = w_3 = w_2$, which is what we had to show. $\square$

Remark 2.7.47. It follows from this lemma that if $k_1, k_2 \in K$ are such that $vk_1 = vk_2$ for some $v \in V^*$, then $k_1 = k_2$, since, by Lemma 2.7.45, there exists a $w \in W$ such that $vk_1 = vw = vk_2$.

Theorem 2.7.48. *$K_{+,\bullet}$ is a commutative field.*

Proof. We have already shown in Lemma 2.7.44 that K is a commutative ring. It only remains to show that every element of K^* is invertible. Let k be an arbitrary non-zero element of K. Since Ω is normal, $\epsilon k = \epsilon w$ for some $w \in W$; hence by Lemma 2.7.46, $k = w$. It follows that k is invertible with inverse $k^{-1} = w^{-1}$, since $w \bullet w^{-1} = w^{-1} \bullet w = \delta$. $\square$

By Lemma 2.7.46, $(K, +) \cong W$ as additive groups. We will denote the isomorphism by square brackets, that is, for every $t \in K$, we will denote the corresponding element of W by $[t]$. Since K is a commutative field, V is a (left) vector space over K, with scalar multiplication given by $tv := v[t]$, for all $t \in K$ and all $v \in V$. From now on, we will denote the multiplicative identity of K by 1 in place of δ. Then $\delta = [1] \in W$. If there is no danger of confusion, we will also write st in place of $s \bullet t$ for $s, t \in K$, and t^2 in place of $t \bullet t$ for $t \in K$.

Definition 2.7.49. We define a map q from V to K by setting $[q(v)] = \delta v = [1]v$ for all $v \in V$. Furthermore, we define a map f from $V \times V$ to K by setting $[f(v_1, v_2)] = F(v_1, v_2)$ for all $v_1, v_2 \in V$.

Lemma 2.7.50. *For all $v \in V$, all $w \in W$ and all $t \in K$, we have that*

(i) $\overline{vw} = \overline{v}w$;

(ii) $\overline{tv} = t\overline{v}$.

Proof. We first show (i). If $v \in \epsilon W$, then $vw \in \epsilon W$ as well, and it follows from Theorem 2.7.19 and Lemma 2.7.44 that $\overline{vw} = vw = \overline{v}w$ (remember that $\sigma = 1$). If $v \notin \epsilon W = \epsilon K$, then we have already shown this in Theorem 2.7.43.

Identity (ii) now follows by substituting $[t]$ for w in (i). $\square$

Lemma 2.7.51. *For all $v \in V$ and all $t \in K$, we have that $[t]v = [tq(v)]$.*

Proof. Let $w := [t] \in W$. We have to show that $wv = w \bullet \delta v$. By $(\mathbf{Q}_{16})$ and Lemma 2.7.50(i), $v^{-1} \cdot wv = \overline{vw} = \overline{v}w$. On the other hand, $v^{-1} \cdot (w \bullet \delta v) = v^{-1} \cdot \delta v \cdot w = \overline{v}w$, again by $(\mathbf{Q}_{16})$. Hence $v^{-1} \cdot wv = v^{-1} \cdot (w \bullet \delta v)$, which implies by Lemma 2.7.46 that $wv = w \bullet \delta v$. $\square$

Lemma 2.7.52. *For all $u, v \in V$ and all $t \in K$, we have that*

$$\pi_u(tv) = t\pi_u(v) \ .$$

Proof. Let $w := [t] \in W$. If we substitute u for v and $\overline{v}(\delta u)^{-1}$ for c in $(\mathbf{Q}_{25})$, then we get that $\pi_u(v)w = \pi_u\big(\overline{\overline{v}(\delta u)^{-1}(wu)}\big)$, and hence, by Lemma 2.7.50(i) and Lemma 2.7.51, that

$$\begin{aligned} t\pi_u(v) &= \pi_u(v)w \\ &= \pi_u(v(\delta u)^{-1}(wu)) \\ &= \pi_u(v \cdot [q(u)^{-1}] \cdot [tq(u)]) \\ &= \pi_u(tq(u)q(u)^{-1}v) \\ &= \pi_u(tv) \ , \end{aligned}$$

which is what we had to show. $\square$

Theorem 2.7.53. *q is an anisotropic quadratic form from V to K with corresponding bilinear form f.*

Proof. Let $v \in V$ and $t \in K$ be arbitrary, and let $w := [t] \in W$. Then, by $(\mathbf{Q}_{26})$,

$$[q(tv)] = [q(vw)] = \delta \cdot vw = \delta \cdot \epsilon w \cdot v = \delta v \cdot \epsilon w \ .$$

By Lemma 2.7.46, it follows from Lemma 2.7.38 that $w \bullet w \bullet z = z \cdot \epsilon w$ for all $z \in W$. Hence

$$[q(tv)] = \delta v \cdot \epsilon w = w \bullet w \bullet \delta v = [t] \bullet [t] \bullet [q(v)] = [t^2 q(v)] \ .$$

Next, it follows from $(\mathbf{Q}_{11})$ that for all $u, v \in V$, $[q(u+v)] = \delta(u+v) = \delta u + \delta v + F(u,v) = [q(u)] + [q(v)] + [f(u,v)] = [q(u) + q(v) + f(u,v)]$. We now show that f is bilinear over K. Let $u, v \in V^*$ and $t \in K$ be arbitrary. By Lemma 2.7.52, we have that $\pi_u(tv) = t\pi_u(v)$. By the definition of π_u, this yields

$$tv - \overline{u^{-1}F(u,tv)} = tv - \overline{tu^{-1}F(u,v)} \ .$$

By Lemma 2.7.50(ii), it follows that $u^{-1}F(u,tv) = tu^{-1}F(u,v)$, hence

$$u^{-1} \cdot [f(u,tv)] = u^{-1} \cdot [f(u,v)] \cdot [t] = u^{-1} \cdot [tf(u,v)] \ .$$

By Lemma 2.7.46, this implies that $f(u,tv) = tf(u,v)$. Since f is symmetric, it follows from this that f is bilinear over K.

Finally, q is anisotropic, since $q(v) = 0$ implies that $\delta v = 0$ and hence $v = 0$. $\square$

Lemma 2.7.54. *For all $u, v \in V$, we have $q(\overline{v}) = q(v)$ and $f(\overline{u}, \overline{v}) = f(u, v)$.*

Proof. We have that

$$\begin{aligned} q(\overline{v}) &= q(f(\epsilon, v)\epsilon - v) \\ &= q(f(\epsilon, v)\epsilon) + q(v) - f(f(\epsilon, v)\epsilon, v) \\ &= f(\epsilon, v)^2 q(\epsilon) + q(v) - f(\epsilon, v) f(\epsilon, v) \\ &= q(v) \ , \end{aligned}$$

and hence

$$\begin{aligned} f(\overline{u}, \overline{v}) &= q(\overline{u} + \overline{v}) - q(\overline{u}) - q(\overline{v}) \\ &= q(u + v) - q(u) - q(v) \\ &= f(u, v) \end{aligned}$$

as well. □

Theorem 2.7.55. $(V, W, \tau_V, \tau_W, \epsilon, \delta) \cong \Omega_Q(K, V, q)$.

Proof. Observe that $q(\epsilon) = 1$, since $[q(\epsilon)] = \delta\epsilon = \delta = [1]$.

Let ϕ be the isomorphism from $[V]$ to V which maps $[v]$ to v for all $v \in V$, and let ψ be the isomorphism from $[K]$ to W which maps $[t]$ to $[t]$ for all $t \in W$. Then $\phi([\epsilon]) = \epsilon$ and $\psi([1]) = [1] = \delta$. Furthermore,

$$\begin{aligned} &\phi([v][t]) = \phi([tv]) = tv = v[t] = \phi([v])\psi([t]) \ , \text{ and} \\ &\psi([t][v]) = \psi([tq(v)]) = [tq(v)] = [t]v = \psi([t])\phi([v]) \ , \end{aligned}$$

for all $t \in K$ and all $v \in V$. Hence (ϕ, ψ) is an isomorphism from $\Omega_Q(K, V, q)$ to $(V, W, \tau_V, \tau_W, \epsilon, \delta)$. □

This completes the proof of Theorem 2.7.7.

2.7.3 Quadrangular Systems of Indifferent Type

Our goal in this section is to classify the quadrangular systems which are indifferent.

Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system which is indifferent. Then $F \equiv 0$ and $H \equiv 0$. By Lemma 2.7.11, all elements of V and W have order 1 or 2, and for all $v \in V$, we have $\overline{v} = v$. Furthermore, we have $\pi_v(c) = c$ for all $v, c \in V$.

Lemma 2.7.56. *K is abelian, i.e. $vk_1k_2 = vk_2k_1$ for all $v \in V$ and all $k_1, k_2 \in K$.*

Proof. By Lemma 2.7.36, we have that $\pi_v(vk) = vk^\sigma$ and hence $vk = vk^\sigma$ for all $v \in V$ and all $k \in K$. It follows that $vk_1k_2 = v(k_1 \bullet k_2) = v(k_1 \bullet k_2)^\sigma = v(k_2^\sigma \bullet k_1^\sigma) = vk_2^\sigma k_1^\sigma = vk_2k_1$ for all $v \in V$ and all $k_1, k_2 \in K$. □

Lemma 2.7.57. *For all $v \in V$ and all $k \in K$, we have that $vkk = v \cdot \delta(\epsilon k)$.*

Proof. It already follows from Lemma 2.7.39 that $vmm = v \cdot \delta(\epsilon m)$ for all $m \in M$. Now suppose that $vk_1k_1 = v \cdot \delta(\epsilon k_1)$ and $vk_2k_2 = v \cdot \delta(\epsilon k_2)$ for some $k_1, k_2 \in K$. We will show that $v(k_1+k_2)(k_1+k_2) = v \cdot \delta(\epsilon(k_1+k_2))$, which will prove the lemma.

By $(\mathbf{Q}_{11})$, $(\mathbf{Q}_{12})$ and Lemma 2.7.56, we have that

$$\begin{aligned} v(k_1+k_2)(k_1+k_2) &= vk_1k_1 + vk_2k_2 + vk_1k_2 + vk_2k_1 \\ &= v \cdot \delta(\epsilon k_1) + v \cdot \delta(\epsilon k_2) + vk_1k_2 + vk_1k_2 \\ &= v \cdot (\delta(\epsilon k_1) + \delta(\epsilon k_2)) \\ &= v \cdot \delta(\epsilon k_1 + \epsilon k_2) \\ &= v \cdot \delta(\epsilon(k_1+k_2)) \,, \end{aligned}$$

and we are done. □

Lemma 2.7.58. *For all $v \in V$ and all $k \in K$, we have that $\delta \cdot vk = \delta \cdot \epsilon k \cdot v$.*

Proof. In Lemma 2.7.15, we have already shown this for all $k \in M$. Now suppose that $\delta \cdot vk_1 = \delta \cdot \epsilon k_1 \cdot v$ and $\delta \cdot vk_2 = \delta \cdot \epsilon k_2 \cdot v$ for some $k_1, k_2 \in K$. We will then show that $\delta \cdot v(k_1+k_2) = \delta \cdot \epsilon(k_1+k_2) \cdot v$, which will prove the lemma.

By $(\mathbf{Q}_{11})$, we have that

$$\begin{aligned} \delta \cdot v(k_1+k_2) &= \delta \cdot (vk_1 + vk_2) \\ &= \delta \cdot vk_1 + \delta \cdot vk_2 \\ &= \delta \cdot \epsilon k_1 \cdot v + \delta \cdot \epsilon k_2 \cdot v \\ &= \delta \cdot (\epsilon k_1 + \epsilon k_2) \cdot v \\ &= \delta \cdot \epsilon(k_1+k_2) \cdot v \,, \end{aligned}$$

and we are done. □

Theorem 2.7.59. *$K_{+,\bullet}$ is a commutative field of characteristic 2 with multiplicative identity δ.*

Proof. We have already shown in Lemma 2.7.56 that $K_{+,\bullet}$ is a commutative ring. Let $k \in K$ be arbitrary. If $\delta(\epsilon k) = 0$, then it would follow from Lemma 2.7.58 that $vk = 0$ for all $v \in V$ and thus $k = 0$. Hence $\delta(\epsilon k)$ is invertible for all $k \neq 0$, and it then follows from Lemma 2.7.57 that $vkk(\delta(\epsilon k))^{-1} = v$ for all $v \in V$. This implies that k is invertible with inverse $k^{-1} := k \bullet (\delta(\epsilon k))^{-1}$.

Furthermore, for all $v \in V$ and all $k \in K$, we have that $v(k+k) = vk + vk = 0$, hence $k + k = 0$, so $\mathrm{char}(K) = 2$. □

Lemma 2.7.60. *If $vk_1 = vk_2$ for some $v \in V^*$ and some $k_1, k_2 \in K$, then $k_1 = k_2$.*

Proof. If $vk_1 = vk_2$ for some $v \in V^*$ and some $k_1, k_2 \in K$, then $v(k_1 + k_2) = 0$. If we would have that $k_1 \neq k_2$, then $k_1 + k_2$ would be invertible, and it would then follow that $v = v(k_1 + k_2)(k_1 + k_2)^{-1} = 0$, a clear contradiction. Hence $k_1 = k_2$. □

Theorem 2.7.61. *$(K, W, \delta V)$ is an indifferent set. Moreover, $\delta v \bullet w = wv$ and $w \bullet w \bullet \delta v = \delta \cdot vw$ for all $v \in V$ and all $w \in W$.*

Proof. It is obvious that W is a subgroup of $(K, +)$. Since $\delta v_1 + \delta v_2 = \delta(v_1 + v_2)$ by $(\mathbf{Q}_{11})$, δV is a subgroup of $(K, +)$ as well. Furthermore, both W and δV contain the multiplicative identity δ.

By $(\mathbf{Q}_{25})$, $\epsilon \cdot \delta v \cdot w = \epsilon \cdot wv$ for all $v \in V$ and all $w \in W$. It follows by Lemma 2.7.60 that $\delta v \bullet w = wv$, and hence $\delta V \bullet W \subseteq W$.

By $(\mathbf{Q}_{26})$, we have that $\delta \cdot vw = \delta \cdot \epsilon w \cdot v = \delta v \cdot \epsilon w$, for all $v \in V$ and all $w \in W$. By Lemma 2.7.60, it follows from Lemma 2.7.38 that $w \bullet w \bullet z = z \cdot \epsilon w$ for all $z \in W$. Hence $\delta v \cdot \epsilon w = w \bullet w \bullet \delta v$. It follows that $w \bullet w \bullet \delta v = \delta \cdot vw$, and hence $W^2 \bullet \delta V \subseteq \delta V$.

Finally, it follows from the definition of K that K is generated by W as a ring. This shows that $(K, W, \delta V)$ is an indifferent set. □

Theorem 2.7.62. $(V, W, \tau_V, \tau_W, \epsilon, \delta) \cong \Omega_D(K, W, \delta V)$.

Proof. First, we observe that v is uniquely determined by δv, since $\delta(v_1 + v_2) = \delta v_1 + \delta v_2$ by $(\mathbf{Q}_{11})$.

Let ϕ be the isomorphism from $[\delta V]$ to V which maps $[\delta v]$ to v for all $v \in V$, and let ψ be the isomorphism from $[W]$ to W which maps $[w]$ to w for all $w \in W$. Then $\phi([\delta]) = \phi([\delta\epsilon]) = \epsilon$ and $\psi([\delta]) = \delta$. Furthermore, it follows from Theorem 2.7.61 that

$$\phi([\delta v][w]) = \phi([w^2 \bullet \delta v]) = \phi([\delta \cdot vw]) = vw = \phi([\delta v])\psi([w]) \text{ , and}$$
$$\psi([w][\delta v]) = \psi([\delta v \bullet w]) = \psi([wv]) = wv = \psi([w])\phi([\delta v]) \text{ ,}$$

for all $v \in V$ and all $w \in W$. Hence (ϕ, ψ) is an isomorphism from $\Omega_D(K, W, \delta V)$ to $(V, W, \tau_V, \tau_W, \epsilon, \delta)$. □

This completes the proof of Theorem 2.7.8, and thereby the classification of all *reduced* quadrangular systems.

2.7.4 Quadrangular Systems of Pseudo-quadratic Form Type, Part I

Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a wide quadrangular system which is the extension of a reduced quadrangular system Λ of proper involutory type, i.e. $\Lambda = (V, \mathrm{Rad}(H), \tau_V, \tau_W, \epsilon, \delta) \cong \Omega_I(K, K_0, \sigma)$ with $K = \langle K_0 \rangle$ and $\sigma \neq 1$, where τ_V and τ_W are as in Remark 2.7.3. In particular, $V = [K]$.

Definition 2.7.63. Let $Y := \mathrm{Rad}(H)$. Note that Y is a normal subgroup of W since $Y \subseteq Z(W)$ by $(\mathbf{Q}_8)$; let $\widetilde{W} := W/Y$. Let ι be the canonical surjection from W to $\widetilde{W}$. We will also write $\tilde{w}$ in place of $\iota(w)$, for all $w \in W$. Then $\tilde{w}_1 = \tilde{w}_2$ if and only if $w_1 \boxminus w_2 \in Y$.

By $(\mathbf{Q}_8)$ and $(\mathbf{Q}_7)$, $[w_1, w_2] \in \mathrm{Im}(F) \subseteq Y$, hence $\widetilde{W}$ is abelian; we will use the additive notations $+$ and $-$ for $\widetilde{W}$. We can define a map $\tilde{\tau}_W$ from $\widetilde{W} \times V$ to $\widetilde{W}$, which we will denote by $\cdot$ or by juxtaposition, by setting

$$\tilde{\tau}_W(\tilde{w}, v) := \tilde{w} \cdot v := \widetilde{wv}$$

for all $v \in V$ and all $w \in W$. This is well defined: let $\tilde{w}_1 = \tilde{w}_2$, then $w_1 \boxminus w_2 \in Y$, and hence $w_1 v \boxminus w_2 v = (w_1 \boxminus w_2)v \in Y$ since $Y \cdot V = Y$; it then follows that $\widetilde{w_1 v} = \widetilde{w_2 v}$.

Remark 2.7.64. If $s \in K_0$, then the notation $[s]$ is ambiguous. If we want to make clear whether we mean $[s] \in V$ or $[s] \in W$, we will write $[s]_V$ and $[s]_W$, respectively. Note that $[s]_W \in Y$ for all $s \in K_0$, and that $\epsilon[s]_W = [s]_V$ for all $s \in K_0$.

Theorem 2.7.65. *$\widetilde{W}$ is a right vector space over K, with scalar multiplication given by $\tilde{w}t := \tilde{w} \cdot [t]$, for all $t \in K$ and all $\tilde{w} \in \widetilde{W}$.*

Proof. We have that

$$\begin{aligned}(\tilde{w}_1 + \tilde{w}_2)t &= (\tilde{w}_1 + \tilde{w}_2) \cdot [t] = \iota\big((w_1 + w_2) \cdot [t]\big) \\ &= \iota(w_1 \cdot [t]) + \iota(w_2 \cdot [t]) = \tilde{w}_1 \cdot [t] + \tilde{w}_2 \cdot [t] = \tilde{w}_1 t + \tilde{w}_2 t\end{aligned}$$

for all $t \in K$ and all $w_1, w_2 \in W$. By $(\mathbf{Q}_{11})$ and $(\mathbf{Q}_7)$,

$$\begin{aligned}\tilde{w}(t_1 + t_2) &= \tilde{w} \cdot [t_1 + t_2] = \tilde{w} \cdot ([t_1] + [t_2]) = \iota(w \cdot ([t_1] + [t_2])) \\ &= \iota(w \cdot [t_1] \boxplus w \cdot [t_2] \boxplus F([t_2]w, [t_1])) \\ &= \iota(w \cdot [t_1]) + \iota(w \cdot [t_2]) = \tilde{w} \cdot [t_1] + \tilde{w} \cdot [t_2] = \tilde{w}t_1 + \tilde{w}t_2\end{aligned}$$

for all $t_1, t_2 \in K$ and all $w \in W$.

It only remains to show that $\tilde{w}(t_1t_2) = (\tilde{w}t_1)t_2$ for all $t_1, t_2 \in K$ and all $w \in W$. (The other axioms for a vector space are obvious.) We thus have to check that $\iota(w \cdot [t_1t_2]) = \iota(w \cdot [t_1] \cdot [t_2])$. Since $K = \langle K_0 \rangle$, it suffices to show this for $t_1 \in K_0$; the result for $t_1 = s_1 \dots s_n$ with $s_1, \dots, s_n \in K_0$ will then follow by induction on n, and since we have already shown that $\tilde{w} \cdot [t_3 + t_4] = \tilde{w} \cdot [t_3] + \tilde{w} \cdot [t_4]$ for all $t_3, t_4 \in K$, the result then follows for all $t_1 \in K$.

By Remark 2.7.64, ($\mathbf{Q}_{26}$) and the definition of $\Omega_I(K, K_0, \sigma)$, we have that

$$\begin{aligned} w \cdot [s]_V \cdot [t] &= w \cdot \epsilon[s]_W \cdot [t] \\ &= w \cdot [t][s]_W \\ &= w \cdot [st]_V \end{aligned}$$

and hence $\iota(w \cdot [s]_V \cdot [t]) = \iota(w \cdot [st]_V)$ for all $s \in K_0$ and all $t \in K$, which is what we had to show. ☐

Definition 2.7.66. Let π be the map from $\widetilde{W}$ to $V/[K_0]_V$ which maps $\tilde{w}$ to $\epsilon w \pmod{[K_0]_V}$. This map is well defined: let $w_1, w_2 \in W$ be such that $\tilde{w}_1 = \tilde{w}_2$. Then $w_1 \boxminus w_2 \in Y$, hence

$$\begin{aligned} \epsilon w_1 - \epsilon w_2 &= \epsilon((w_1 \boxminus w_2) \boxplus w_2) - \epsilon w_2 \\ &= \epsilon(w_1 \boxminus w_2) + \epsilon w_2 - \epsilon w_2 \\ &= \epsilon(w_1 \boxminus w_2) \in \epsilon Y = \epsilon[K_0]_W = [K_0]_V \end{aligned}$$

by ($\mathbf{Q}_{12}$).

Lemma 2.7.67. *For all $\tilde{w} \in \widetilde{W}$, we have that $\pi(\tilde{w}) = 0$ if and only if $\tilde{w} = 0$.*

Proof. Let $w \in W$ be such that $\pi(\tilde{w}) = 0$. Then $\epsilon w \in [K_0]_V$, say $\epsilon w = [s]_V$ with $s \in K_0$. By 2.2.13(ii), $\epsilon(\boxminus [s]_W) = -\epsilon[s]_W$ since $[s]_W \in Y$. It follows that $\epsilon(w \boxminus [s]_W) = \epsilon w + \epsilon(\boxminus [s]_W) = \epsilon w - \epsilon[s]_W = \epsilon w - [s]_V = 0$, and hence $w = [s]_W \in Y$. It follows that $\tilde{w} = 0$. ☐

Definition 2.7.68. Let h be the map from $\widetilde{W} \times \widetilde{W}$ to K, defined by the identity $[h(\tilde{w}_1, \tilde{w}_2)] := H(w_1, w_2)$ for all $\tilde{w}_1, \tilde{w}_2 \in \widetilde{W}$. Since $\widetilde{W} = W/\operatorname{Rad}(H)$, the map h is well defined.

Lemma 2.7.69. *For all $v \in V$, all $w \in W$ and all $y_1, \dots, y_n \in Y$, we have that $\overline{\overline{v}w y_1 \dots y_n} = \overline{\overline{v y_1 \dots y_n} w}$.*

Proof. We will first prove the lemma for $n = 1$. Let $v \in V$, $w \in W$ and $y \in Y$. We may assume that $w \neq 0$. Observe that $\Pi_w(y) = y$ since $y \in \mathrm{Rad}(H)$. It follows from $(\mathbf{Q}_{23})$ that $\overline{vy} = -\overline{\overline{\overline{v}(\boxminus w)}y}\kappa(w)$, and hence, by $(\mathbf{Q}_{18})$, $\overline{\overline{vy}(\boxminus w)} = \overline{\overline{v}(\boxminus w)}y$. Substituting $\boxminus w$ for w now yields the result for $n = 1$.

We advance to general n by induction. Let $v \in V$, let $w \in W$ and let $y_1, \dots, y_n \in Y$, and suppose that $\overline{\overline{v}w}y_1 \dots y_{n-1} = \overline{\overline{vy_1 \dots y_{n-1}}w}$. Then

$$\begin{aligned}\overline{\overline{v}w}y_1 \dots y_n &= \overline{\overline{vy_1 \dots y_{n-1}}w} \cdot y_n \\ &= \overline{\overline{vy_1 \dots y_{n-1}y_n}w}\,,\end{aligned}$$

where we have used the lemma for $n = 1$ in the last equality. This completes the proof of this lemma. $\square$

Lemma 2.7.70. *For all $w \in W$ and all $y_1, \dots, y_n \in Y$, we have that*

$$\overline{\epsilon w}y_1 \dots y_n = \overline{\overline{\epsilon y_n \dots y_1}w}\,.$$

In particular, we have that $\overline{[s]_V w} = \overline{\epsilon w}[s]_W$ for all $w \in W$ and all $s \in K_0$.

Proof. By Theorem 2.7.19, $\overline{\epsilon y_1 \dots y_n} = \epsilon y_n \dots y_1$. The first result follows by substituting ϵ for v in Lemma 2.7.69.

In the particular case $n = 1$ and $y_1 = [s]_W$, we get that $\overline{\epsilon w}[s]_W = \overline{\epsilon[s]_W w} = \overline{[s]_V w}$. $\square$

Theorem 2.7.71. *The map h is a skew-hermitian form over K with respect to σ.*

Proof. Since $\overline{H(w_1, w_2)} = -H(w_2, w_1)$ for all $w_1, w_2 \in W$ by $(\mathbf{Q}_{22})$, and since $\overline{[t]} = [t^\sigma]$ for all $t \in K$, it follows that $h(\tilde{w}_1, \tilde{w}_2) = -h(\tilde{w}_2, \tilde{w}_1)^\sigma$, for all $\tilde{w}_1, \tilde{w}_2 \in \widetilde{W}$.

By $(\mathbf{Q}_{22})$, $(\mathbf{Q}_{12})$ and Lemma 2.7.70,

$$\begin{aligned}H(w_1 \cdot [s]_V, w_2) &= -\overline{H(w_2, w_1 \cdot [s]_V)} \\ &= -\overline{[s]_V \cdot (w_1 \boxplus w_2)} + \overline{[s]_V \cdot w_1} + \overline{[s]_V \cdot w_2} \\ &= -\overline{\epsilon(w_1 \boxplus w_2)}[s]_W + \overline{\epsilon w_1}[s]_W + \overline{\epsilon w_2}[s]_W \\ &= -\overline{H(w_2, w_1)} \cdot [s]_W \\ &= H(w_1, w_2) \cdot [s]_W\,,\end{aligned}$$

for all $w_1, w_2 \in W$ and all $s \in K_0$. Hence

$$[h(\tilde{w}_1 s, \tilde{w}_2)] = [h(\tilde{w}_1, \tilde{w}_2)] \cdot [s]_W = [sh(\tilde{w}_1, \tilde{w}_2)]\,,$$

from which it follows that $h(\tilde{w}_1 s, \tilde{w}_2) = sh(\tilde{w}_1, \tilde{w}_2)$ for all $w_1, w_2 \in W$ and all $s \in K_0$. Since $K = \langle K_0 \rangle$ and since h is additive in both variables, it follows from this that $h(\tilde{w}_1 t, \tilde{w}_2) = t^\sigma h(\tilde{w}_1, \tilde{w}_2)$ for all $w_1, w_2 \in W$ and all $t \in K$.

Finally,

$$\begin{aligned} h(\tilde{w}_1, \tilde{w}_2 t) &= -h(\tilde{w}_2 t, \tilde{w}_1)^\sigma = -(t^\sigma h(\tilde{w}_2, \tilde{w}_1))^\sigma \\ &= -h(\tilde{w}_2, \tilde{w}_1)^\sigma t = h(\tilde{w}_1, \tilde{w}_2) t \end{aligned}$$

for all $w_1, w_2 \in W$ and all $t \in K$. This shows that h is a skew-hermitian form over K with respect to σ. □

Definition 2.7.72. For all $t \in K$, let k_t be the homomorphism from V to itself which maps $[t']$ to $[tt']$ for all $[t'] \in [K] = V$. We denote the action of k_t by right juxtaposition, i.e. $[t']k_t = [tt']$ for all $t, t' \in K$. In particular, we can identify k_s and $[s]_W$ for all $s \in K_0$. Moreover, we set $k_t^\sigma := k_{t^\sigma}$ for all $t \in K$. Note that the set $\{k_t \mid t \in K\}$ coincides with the set K that we defined in the beginning of section 2.7.1. In particular, we can apply the lemmas and theorems of that section on the sub-quadrangular system $\Lambda = (V, Y, \tau_V, \tau_W, \epsilon, \delta)$.

Lemma 2.7.73. *For all $w \in W$ and all $t_1, t_2 \in K$, we have that*

$$\epsilon k_1 w k_2^\sigma + \epsilon k_2 w k_1^\sigma = \epsilon F([t_2]w, [t_1]) + H(w[t_2], w[t_1]) \ ,$$

where $k_1 := k_{t_1}$ and $k_2 := k_{t_2}$.

Proof. By Lemma 2.7.22 and the definition of the map $v \mapsto \overline{v}$, we have that

$$\begin{aligned} \epsilon F([t_2]w, [t_1]) &= \epsilon F(\epsilon k_2 w, \epsilon k_1) \\ &= \epsilon F(\epsilon k_2 w k_1^\sigma, \epsilon) \\ &= \epsilon k_2 w k_1^\sigma + \overline{\epsilon k_2 w k_1^\sigma} \ , \end{aligned}$$

and by Theorem 2.7.71 and 2.2.13(ii), we have that

$$\begin{aligned} H(w[t_2], w[t_1]) &= [h(\tilde{w} t_2, \tilde{w} t_1)] \\ &= [t_2^\sigma h(\tilde{w}, \tilde{w} t_1)] \\ &= [h(\tilde{w}, \tilde{w} t_1)] k_2^\sigma \\ &= H(w, w[t_1]) k_2^\sigma \\ &= [t_1](\boxminus w) k_2^\sigma + [t_1] w k_2^\sigma \\ &= \epsilon k_1 (\boxminus w) k_2^\sigma + \epsilon k_1 w k_2^\sigma \ . \end{aligned}$$

It only remains to show that $\overline{\epsilon k_2 w k_1^\sigma} = -\epsilon k_1(\boxminus w)k_2^\sigma$. By Lemma 2.7.70, we have that $\epsilon k_t^\sigma w = \overline{\overline{\epsilon w} k_t}$ for all $w \in W$ and all $t \in K$. By ($\mathbf{Q}_{16}$) with ϵ in place of v and Lemma 2.7.18 with $\epsilon(\boxminus w)$ in place of v, it follows that

$$\begin{aligned}\overline{\epsilon k_2 w k_1^\sigma} &= \overline{\overline{\overline{\epsilon w} k_2^\sigma} k_1^\sigma} \\ &= -\overline{\overline{\epsilon(\boxminus w) k_2^\sigma} k_1^\sigma} \\ &= -\overline{\overline{\epsilon(\boxminus w)} k_1^\sigma k_2^\sigma} \\ &= -\epsilon k_1(\boxminus w)k_2^\sigma \,,\end{aligned}$$

which completes the proof of this lemma. □

Theorem 2.7.74. *For all $w \in W$ and all $t \in K$, we have that*

$$\epsilon \cdot w[t] = \epsilon k_t w k_t^\sigma \,.$$

Proof. First assume that $t = s_1 \dots s_n$ with $s_1, \dots, s_n \in K_0$. For all $i \in \{1, \dots, n\}$, let $y_i := [s_i]_W \in Y$. Then we have to show that $\epsilon \cdot w(\epsilon y_1 \dots y_n) = \epsilon y_1 \dots y_n w y_n \dots y_1$ for all $w \in W$. By ($\mathbf{Q}_{16}$), Lemma 2.7.27 and Lemma 2.7.70,

$$\begin{aligned}\epsilon y_1 \dots y_n w &= -\overline{(\epsilon y_1 \dots y_n)^{-1}(\boxminus w \cdot \epsilon y_1 \dots y_n)} \\ &= -\overline{\epsilon y_n^{-1} \dots y_1^{-1}(\boxminus w \cdot \epsilon y_1 \dots y_n)} \\ &= -\overline{\epsilon(\boxminus w \cdot \epsilon y_1 \dots y_n)} y_1^{-1} \dots y_n^{-1} \\ &= \epsilon(w \cdot \epsilon y_1 \dots y_n) y_1^{-1} \dots y_n^{-1} \,,\end{aligned}$$

from which it follows that $\epsilon(w \cdot \epsilon y_1 \dots y_n) = \epsilon y_1 \dots y_n w y_n \dots y_1$.

Now suppose that $\epsilon \cdot w[t_1] = \epsilon k_1 w k_1^\sigma$ and $\epsilon \cdot w[t_2] = \epsilon k_2 w k_2^\sigma$ for some $t_1, t_2 \in K$, where where $k_1 := k_{t_1}$ and $k_2 := k_{t_2}$. We will show that

$$\epsilon \cdot w[t_1 + t_2] = \epsilon(k_1 + k_2)w(k_1 + k_2)^\sigma \,,$$

which will prove the theorem for all $t \in K$, since $K = \langle K_0 \rangle$.

By ($\mathbf{Q}_{11}$), ($\mathbf{Q}_{12}$) with $v = \epsilon$, ($\mathbf{Q}_7$) and Lemma 2.7.73, we have that

$$\begin{aligned}\epsilon \cdot w[t_1 + t_2] &= \epsilon \cdot w([t_1] + [t_2]) \\ &= \epsilon \cdot (w[t_1] \boxplus w[t_2] \boxplus F([t_2]w, t_1)) \\ &= \epsilon \cdot w[t_1] + \epsilon \cdot w[t_2] + \epsilon F([t_2]w, t_1) + H(w[t_2], w[t_1]) \\ &= \epsilon k_1 w k_1^\sigma + \epsilon k_2 w k_2^\sigma + \epsilon k_1 w k_2^\sigma + \epsilon k_2 w k_1^\sigma \\ &= \epsilon(k_1 + k_2)w(k_1 + k_2)^\sigma \,,\end{aligned}$$

which completes the proof of this theorem. □

Lemma 2.7.75. *Let $w \in W$ and $t \in K$ be arbitrary. Let $x \in K$ be such that $\epsilon w = [x]$. Then $[t]w = [xt]$, and $\epsilon k_t w k_t^\sigma = [t^\sigma x t]$.*

Proof. By Lemma 2.7.70, we have that

$$[t]w = \epsilon k_t w = \overline{\overline{\epsilon w} k_t^\sigma} = \overline{[x^\sigma] k_t^\sigma} = \overline{[t^\sigma x^\sigma]} = [xt] \ ;$$

it then follows that

$$\epsilon k_t w k_t^\sigma = [xt] k_t^\sigma = [t^\sigma x t] \ ,$$

and we are done. $\square$

Definition 2.7.76. For all $\tilde{w} \in \widetilde{W}$, let $p(\tilde{w})$ be any element $t \in K$ such that $[t]$ is contained in the coset $\pi(\tilde{w}) \in V/[K_0]_v$. Hence p is a map from $\widetilde{W}$ to K such that $[p(\tilde{w})] \equiv \epsilon w \pmod{[K_0]_v}$.

Theorem 2.7.77. $(K, K_0, \sigma, \widetilde{W}, p)$ *is an anisotropic pseudo-quadratic space with corresponding skew-hermitian form h.*

Proof. It only remains to show that p is a pseudo-quadratic form. All equivalences will be modulo $[K_0]_v$. By $(\mathbf{Q}_{12})$, we have that

$$\begin{aligned}
[p(\tilde{w}_1 + \tilde{w}_2)] &\equiv [p(\tilde{w}_2 + \tilde{w}_1)] \\
&\equiv \epsilon(w_2 \boxplus w_1) \\
&\equiv \epsilon w_2 + \epsilon w_1 + H(w_1, w_2) \\
&\equiv [p(\tilde{w}_1)] + [p(\tilde{w}_2)] + [h(\tilde{w}_1, \tilde{w}_2)]
\end{aligned}$$

for all $w_1, w_2 \in W$, which shows the first property.

Next, let w be an arbitrary element of W, and let $x \in K$ be such that $\epsilon w = [x]$. Then $[p(\tilde{w})] \equiv [x]$, and hence $[t^\sigma p(\tilde{w}) t] \equiv [t^\sigma x t]$ as well, since $t^\sigma K_0 t \subseteq K_0$. It follows from Theorem 2.7.74 and Lemma 2.7.75 that

$$[p(\tilde{w}t)] \equiv \epsilon \cdot w[t] \equiv \epsilon k_t w k_t^\sigma \equiv [t^\sigma x t] \equiv [t^\sigma p(\tilde{w}) t] \ ,$$

which shows the second property.

Finally, if $[p(\tilde{w})] \equiv 0$ for some $\tilde{w} \in \widetilde{W}$, then $\pi(\tilde{w}) = 0$ and hence $\tilde{w} = 0$ by Lemma 2.7.67. $\square$

Lemma 2.7.78. *Let the group T be as in section 2.6.4. For each element $(a, x) \in T$, there is a unique element $w \in W$ such that $w \in a$ and $\epsilon w = [x]$. If we denote this element by $\chi(a, x)$, then χ is an isomorphism from T to W.*

Proof. Let $(a, x) \in T$ be arbitrary. Choose an arbitrary element $z \in a$. Then $a = \tilde{z}$, and $\epsilon z \equiv [x] \pmod{[K_0]_V}$ by the definition of T. Hence $\epsilon z - [x] \in [K_0]_V = \epsilon Y$, say $\epsilon z - [x] = \epsilon y$ with $y \in Y$. Set $w = z \boxminus y$, then $\tilde{w} = \tilde{z} = a$, and $\epsilon w = \epsilon(z \boxminus y) = \epsilon z - \epsilon y = [x]$ by ($\mathbf{Q}_{12}$) and 2.2.13(ii). This shows the existence of w.

Now suppose that $w_1, w_2 \in W$ are such that $\tilde{w}_1 = \tilde{w}_2$ and $\epsilon w_1 = \epsilon w_2$. Then $w_1 \boxminus w_2 \in Y$, and hence, by ($\mathbf{Q}_{12}$),

$$\begin{aligned} 0 &= \epsilon w_1 - \epsilon w_2 \\ &= \epsilon((w_1 \boxminus w_2) \boxplus w_2) - \epsilon w_2 \\ &= \epsilon(w_1 \boxminus w_2) + \epsilon w_2 - \epsilon w_2 \\ &= \epsilon(w_1 \boxminus w_2) , \end{aligned}$$

from which it follows that $w_1 = w_2$.

Hence $\chi : T \to W$ is a well defined map, which is bijective, with the inverse map given by $\chi^{-1}(w) = (\tilde{w}, x) \in T$, where $[x] = \epsilon w$. In order to show that χ is an isomorphism, it now suffices to show that $\chi^{-1}(w_1 \boxplus w_2) = \chi^{-1}(w_1) \boxplus \chi^{-1}(w_2)$. Let $x_1, x_2 \in K$ be such that $[x_1] = \epsilon w_1$ and $[x_2] = \epsilon w_2$. Then $\epsilon(w_1 \boxplus w_2) = \epsilon w_1 + \epsilon w_2 + H(w_2, w_1) = [x_1] + [x_2] + [h(\tilde{w}_2, \tilde{w}_1)]$, and hence

$$\begin{aligned} \chi^{-1}(w_1 \boxplus w_2) &= (\tilde{w}_1 + \tilde{w}_2, x_1 + x_2 + h(\tilde{w}_2, \tilde{w}_1)) \\ &= (\tilde{w}_1, x_1) \boxplus (\tilde{w}_2, x_2) \\ &= \chi^{-1}(w_1) \boxplus \chi^{-1}(w_2) , \end{aligned}$$

which completes the proof of this lemma. □

Theorem 2.7.79. $(V, W, \tau_V, \tau_W, \epsilon, \delta) \cong \Omega_P(K, K_0, \sigma, \widetilde{W}, p)$.

Proof. Let ϕ be the isomorphism from $[K]$ to V which maps $[t]$ to $[t]$ for all $t \in K$, and let ψ be the isomorphism from $[T]$ to W which maps $[a, x]$ to $\chi(a, x)$ for all $(a, x) \in T$. Then $\phi([1]) = [1] = \epsilon$ and $\psi([0, 1]) = \delta$ since $\delta \in Y$ (hence $\tilde{\delta} = 0$) and $\epsilon\delta = [1]$.

Now, let $t \in K$ and $(a, x) \in T$ be arbitrary. Let $w = \psi([a, x]) = \chi(a, x)$, then $a = \tilde{w}$ and $\epsilon w = [x]$. By Lemma 2.7.75, $[xt] = [t]w$, hence

$$\begin{aligned} &\phi([t][a, x]) = \phi([xt]) = [xt] = [t]w = \phi([t])\psi([a, x]) , \text{ and} \\ &\psi([a, x][t]) = \psi([at, t^\sigma xt]) = w[t] = \psi([a, x])\phi([t]) , \end{aligned}$$

since $\widetilde{w[t]} = \tilde{w}t = at$ and $\epsilon \cdot w[t] = [t^\sigma xt]$ by Theorem 2.7.74 and Lemma 2.7.75. This shows that (ϕ, ψ) is an isomorphism from $\Omega_P(K, K_0, \sigma, \widetilde{W}, p)$ to $(V, W, \tau_V, \tau_W, \epsilon, \delta)$. □

This completes the proof of Theorem 2.7.9.

2.7.5 Quadrangular Systems of Type F_4

Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a wide quadrangular system which is the extension of a reduced quadrangular system Λ of quadratic form type, i.e. $\Lambda = (V, \mathrm{Rad}(H), \tau_V, \tau_W, \epsilon, \delta) \cong \Omega_Q(K, V_0, q)$, where τ_V and τ_W are as in Remark 2.7.3.

Our goal in this section is to classify these quadrangular systems in the case that $\mathrm{Rad}(F) \neq 0$. So assume that $\mathrm{Rad}(F) \neq 0$. It then follows from $(\mathbf{Q}_{10})$ that $\epsilon \in \mathrm{Rad}(F)$. Note that $\overline{v} = v$ for all $v \in V$ by Lemma 2.7.11.

Remark 2.7.80. We will identify V and V_0 in the sequel if there is no danger of confusion, which will allow us to use notations like tv with $t \in K$ and $v \in V$.

Observe that the axiom system is very symmetrical now (see section 2.8.3). In particular, every identity will have a "*dual identity*", which is obtained by switching the roles of V and W.

Lemma 2.7.81. *For all $v, v_1, v_2 \in V$ and all $w, w_1, w_2 \in W$, we have that*

(i) $F(v_1 w, v_2) = F(v_1, v_2 w)$;

(ii) $H(w_1 v, w_2) = H(w_1, w_2 v)$.

Proof. Since both V and W are abelian, it follows from $(\mathbf{Q}_{21})$ and $(\mathbf{Q}_{11})$ that $F(v_1 w, v_2) = w(v_2 + v_1) + wv_2 + wv_1 = w(v_1 + v_2) + wv_1 + wv_2 = F(v_2 w, v_1) = F(v_1, v_2 w)$, which proves (i). Similarly, (ii) follows from $(\mathbf{Q}_{22})$ and $(\mathbf{Q}_{12})$. (Identity (ii) is the "dual" of identity (i).) $\square$

Definition 2.7.82. Let $R := \mathrm{Rad}(F)$. Then $\epsilon \in R$, and $R \neq V$ since $F \not\equiv 0$. Moreover, let $L := q(R) \subseteq K$.

Lemma 2.7.83. *$\Sigma := (R, W, \tau_V, \tau_W, \epsilon, \delta)$ is a quadrangular system with $F_\Sigma \equiv 0$ and $H_\Sigma \not\equiv 0$; see Remark 2.7.3.*

Proof. First of all, we observe that $R = \mathrm{Rad}(F)$ is a subgroup of V, since F is additive in both variables. We have that $\mathrm{Im}(F_\Sigma) = F(R, R) = 0$, and $\mathrm{Im}(H_\Sigma) = H(W, W) = \mathrm{Im}(H) \neq 0$. It now only remains to show that $\tau_V(R \times W) \subseteq R$, $H(W, W) \subseteq R$ and $(R^*)^{-1} \subseteq R$.

If $v \in R$, then $F(v, V) = 0$, hence $F(vw, V) = F(v, Vw) \subseteq F(v, V) = 0$ as well for all $w \in W$, by Lemma 2.7.81(i). Hence $\tau_V(R \times W) = R \cdot W \subseteq R$. Since W is abelian, it follows from $(\mathbf{Q}_8)$ that $H(W, W) \subseteq \mathrm{Rad}(F) = R$. Finally, if $v \in R^*$, then $F(v^{-1}, V) = F(v, V)v^{-1} = 0$ by $(\mathbf{Q}_{17})$, and hence $v^{-1} \in R$. Thus $\Sigma := (R, W, \tau_V, \tau_W, \epsilon, \delta)$ is a quadrangular system. $\square$

By Theorem 2.7.13, $\Sigma^* = (W, R, \tau_W, \tau_V, \delta, \epsilon)$ is a reduced quadrangular system. Suppose that Σ^* were of involutory type, say $\Sigma^* \cong Q_I(J, J_0, \sigma)$ for some involutory set (J, J_0, σ). Then $\delta \in \mathrm{Rad}(H)$ would imply that $0 = F_{\Sigma^*}([1],[a]) = [a + a^\sigma]$ and hence $a + a^\sigma = 0$ for all $a \in J$, from which it would follow that $F_{\Sigma^*}([a],[b]) = [(a^\sigma b) + (a^\sigma b)^\sigma] = 0$ for all $a, b \in J$. Hence $H_\Sigma \equiv F_{\Sigma^*} = 0$, a contradiction.

It follows that Σ^* must be of quadratic form type. In particular, R has the structure of a field, W is a (right) vector space over R, and the map $p : W \to R : w \mapsto \epsilon w$ is a quadratic form. If we denote the multiplication in R by $\bullet$, then we have that $w(r_1 \bullet r_2) = (wr_1)r_2$ for all $w \in W$ and all $r_1, r_2 \in R$.

Lemma 2.7.84. *L is a subfield of K, with $K^2 \subseteq L \subseteq K$. Moreover, q is a field isomorphism from R to L.*

Proof. We will first prove that q is an isomorphism (both additive and multiplicative) from R to L. Since $L = q(R)$, q is surjective. For all $r_1, r_2 \in R$, we have that $[q(r_1 + r_2)] = \delta(r_1 + r_2) = \delta r_1 + \delta r_2 = [q(r_1) + q(r_2)]$, by $(\mathbf{Q}_{11})$; hence q is additive. In particular, if $q(r_1) = q(r_2)$, then $q(r_1 + r_2) = 0$ and hence $r_1 = r_2$, since q is anisotropic. Hence q is injective. Furthermore, for all $r_1, r_2 \in R$, we have that $[q(r_1 \bullet r_2)] = \delta(r_1 \bullet r_2) = (\delta r_1)r_2 = [q(r_1)]r_2 = [q(r_1)q(r_2)]$ by Lemma 2.7.51, hence q is multiplicative.

It follows that $L = q(R)$ is a commutative field which is isomorphic to R. Finally, for all $t \in K$, we have that $q(t\epsilon) = t^2 q(\epsilon) = t^2$, hence $K^2 \subseteq q(R) = L$ since $t\epsilon \in \mathrm{Rad}(F)$. □

Definition 2.7.85. For all $s \in L$, we let $[s] := q^{-1}(s) \in V$. If we want to make clear whether we mean $[s] \in V$ or $[s] \in W$, we will write $[s]_V$ and $[s]_W$, respectively. By Lemma 2.7.84, we can consider W as a (left) vector space over L via the scalar multiplication $sw := w[s]$ for all $w \in W$ and all $s \in L$.

Definition 2.7.86. Let $\hat{q}$ be the map from W to L given by $\hat{q}(w) := q(p(w)) = q(\epsilon w)$ for all $w \in W$, and let $\hat{f}$ be the map from $W \times W$ to L given by $\hat{f}(w_1, w_2) := q(H(w_1, w_2))$ for all $w_1, w_2 \in W$. In particular, $\epsilon w = [\hat{q}(w)]_V$ and $H(w_1, w_2) = [\hat{f}(w_1, w_2)]_V$ for all $w, w_1, w_2 \in W$.

Lemma 2.7.87. *$\hat{q}$ is a quadratic form from W to L with corresponding bilinear form $\hat{f}$.*

Proof. Since p is a quadratic form from W to R with corresponding bilinear form H, it follows by Lemma 2.7.84 that $\hat{q} = q \circ p$ is a quadratic form from W to L with corresponding bilinear form $\hat{f} = q \circ H$. □

Remark 2.7.88. For $s \in L$ and $t \in K$, we will write $q[s]$ and $\hat{q}[t]$ in place of $q([s])$ and $\hat{q}([t])$, respectively.

Lemma 2.7.89. *For all $v, v_1, v_2 \in V$ and all $w, w_1, w_2 \in W$, we have that*

(i) $F(v_1, v_2) = 0 \implies wv_1v_2 = wv_2v_1$;

(ii) $H(w_1, w_2) = 0 \implies vw_1w_2 = vw_2w_1$.

Proof. Observe that ($\mathbf{Q}_{23}$) can be rewritten as "$v\Pi_w(z) = vwzw^{-1}$", and that ($\mathbf{Q}_{24}$) can be rewritten as "$w\pi_v(c) = wvcv^{-1}$" since $\pi_v(\epsilon) = \epsilon$.

Let $v_1, v_2 \in V$ be such that $F(v_1, v_2) = 0$, and assume that $v_1 \neq 0$. Then $\pi_{v_1}(v_2) = v_2$. It then follows from ($\mathbf{Q}_{24}$) that $wv_2 = wv_1v_2v_1^{-1}$ for all $w \in W$, hence (i). Identity (ii) is the dual of (i). □

In particular, $s(wv) = wv[s] = w[s]v = (sw)v$ and $t(vw) = vw[t] = v[t]w = (tv)w$ for all $v \in V$, $w \in W$, $s \in L$ and $t \in K$. It follows that the notations swv and tvw are unambiguous.

Lemma 2.7.90. *For all $v \in V^*$ and all $w \in W^*$, we have that*

(i) $v^{-1} = q(v)^{-1}v$;

(ii) $w^{-1} = \hat{q}(w)^{-1}w$.

Proof. If we substitute δ for w in ($\mathbf{Q}_{16}$), then we get that $v = v^{-1} \cdot \delta v = v^{-1}[q(v)]_W = q(v)v^{-1}$, which proves (i). Similarly for (ii). □

Lemma 2.7.91. *For all $v \in V^*$, $w \in W^*$, $t \in K$ and $s \in L$, we have that*

(i) $w \cdot tv = \hat{q}[t]wv$;

(ii) $v \cdot sw = q[s]vw$;

(iii) $wv = \hat{q}[q(v)]wv^{-1}$;

(iv) $vw = q[\hat{q}(w)]vw^{-1}$.

Proof. We only prove (i) and (iii). By ($\mathbf{Q}_{26}$),

$$w \cdot tv = w \cdot v[t] = w \cdot \epsilon[t] \cdot v = w \cdot [\hat{q}[t]]_V \cdot v = \hat{q}[t]wv \,,$$

which proves (i). It now follows from Lemma 2.7.90(i) and (i) that $wv = w \cdot q(v)v^{-1} = \hat{q}[q(v)]wv^{-1}$, which proves (iii). □

Lemma 2.7.92. *For all $v, c \in V$, $w, z \in W$, we have that*

(i) $wvcv = w(f(v,c)v + q(v)c)$;

(ii) $vwzw = v(\hat{f}(w,z)w + \hat{q}(w)z)$.

Proof. We only prove (i). We may assume that $v \neq 0$. By ($\mathbf{Q}_{24}$) and by Lemma 2.7.90(i),

$$wvcv^{-1} = w\pi_v(c) = w(c + f(v,c)v^{-1}) = w(c + f(v,c)q(v)^{-1}v) \; .$$

It follows by Lemma 2.7.91(iii) and Lemma 2.7.91(i) that

$$\begin{aligned} wvcv &= \hat{q}[q(v)]wvcv^{-1} = \hat{q}[q(v)]w(c + f(v,c)q(v)^{-1}v) \\ &= w(q(v)c + f(v,c)v) \; , \end{aligned}$$

which is what we had to show. □

Lemma 2.7.93. *For all $v, c \in V$, $w, z \in W$, we have that*

(i) $wvc + wcv = \hat{q}[f(v,c)]w + [f(v,c)f(vw,c)]$ *;*

(ii) $vwz + vzw = q[\hat{f}(w,z)]v + [\hat{f}(w,z)\hat{f}(wv,z)]$.

Proof. Again, we only prove (i). We may assume that $v \neq 0$. By ($\mathbf{Q}_{24}$), Lemma 2.7.91(i) and Lemma 2.7.90(i),

$$\begin{aligned} wvcv^{-1} &= w\pi_v(c) \\ &= w(c + f(v,c)v^{-1}) \\ &= wc + w \cdot f(v,c)v^{-1} + F(f(v,c)v^{-1}, cw) \\ &= wc + \hat{q}[f(v,c)]wv^{-1} + [f(f(v,c)q(v)^{-1}v, cw)] \; , \end{aligned}$$

and hence, by Lemma 2.7.51,

$$\begin{aligned} wvc &= wcv + \hat{q}[f(v,c)]w + [f(v,c)q(v)^{-1}f(v,cw)]v \\ &= wcv + \hat{q}[f(v,c)]w + [f(v,c)f(v,cw)] \\ &= wcv + \hat{q}[f(v,c)]w + [f(v,c)f(vw,c)] \; , \end{aligned}$$

which is what we had to show. □

Lemma 2.7.94. *For all $v \in V$, $c \in V^*$, $w \in W$ and $z \in W^*$, we have that*

(i) $z \cdot vz = \hat{q}(z)zv$ *;*

(ii) $c \cdot wc = q(c)cw$ *;*

(iii) $\hat{f}(z, w \cdot vz)z^{-1} = \hat{f}(zv, w)z$ *;*

(iv) $f(c, v \cdot wc)c^{-1} = f(cw, v)c$.

Proof. We will only prove (i) and (iii). First of all, observe that it follows from ($\mathbf{Q}_{12}$) that $H(w, wv) = 0$ for all $v \in V$ and all $w \in W$. In particular, $\Pi_z(z \cdot \epsilon z) = z \cdot \epsilon z$ and $\Pi_z(z \cdot vz) = z \cdot vz$. It thus follows from ($\mathbf{Q}_{26}$) that

$z \cdot vz = z \cdot \epsilon z \cdot v = \hat{q}(z)zv$, which shows (i). By Lemma 2.7.81(ii) and Lemma 2.7.90(ii), it now follows that

$$\begin{aligned} \hat{f}(z, w \cdot vz)z^{-1} &= \hat{f}(z \cdot vz, w)\hat{q}(z)^{-1}z \\ &= \hat{f}(\hat{q}(z)zv, w)\hat{q}(z)^{-1}z \\ &= \hat{f}(zv, w)z \ , \end{aligned}$$

which shows (iii). □

Lemma 2.7.95. *For all $v \in V$ and all $w \in W$, we have that*

(i) $[\hat{q}(wv)] = q(v)[\hat{q}(w)]$;

(ii) $[q(vw)] = \hat{q}(w)[q(v)]$.

Proof. We will only prove (i). We may again assume that $v \neq 0$. Since $\epsilon \in \mathrm{Rad}(F)$, it follows by Lemma 2.7.81(i) that $F(v, \epsilon c) = F(vc, \epsilon) = 0$ and hence $\pi_v(\epsilon c) = \epsilon c$ for all $c \in V$. If we set $c = \epsilon$ in ($\mathbf{Q}_{25}$), we thus get that $\epsilon \cdot \delta v \cdot w = \epsilon \cdot wv$, and hence $[\hat{q}(wv)] = \epsilon \cdot wv = \epsilon \cdot \delta v \cdot w = \epsilon[q(v)]w = q(v)\epsilon w = q(v)[\hat{q}(w)]$. □

Lemma 2.7.96. *For all $v, c \in V$ and all $w, z \in W$, we have that*

(i) $w \cdot vz + \hat{q}(z)wv = \hat{f}(w, zv)z + \hat{f}(w, z)zv$;

(ii) $v \cdot wc + q(c)vw = f(v, cw)c + f(v, c)cw$.

Proof. We will only prove (i). We may assume that $z \neq 0$. By ($\mathbf{Q}_{26}$), we have that

$$w \cdot \epsilon z \cdot v + z^{-1}H(z, w \cdot \epsilon z) \cdot v = w \cdot vz + z^{-1}H(z, w \cdot vz) \ ,$$

hence

$$\hat{q}(z)wv + \hat{f}(z, w \cdot \epsilon z)z^{-1}v = w \cdot vz + \hat{f}(z, w \cdot vz)z^{-1} \ ,$$

and it follows from Lemma 2.7.94(iii) that

$$\hat{q}(z)wv + \hat{f}(z, w)zv = w \cdot vz + \hat{f}(zv, w)z \ ,$$

which is what we had to show. □

At this point, we will break the symmetry. We cannot avoid this, since L is a subfield of K, but not vice versa.

Lemma 2.7.97. *For all $t \in K$ and all $s \in L$, we have that*

(i) $s[t] = [st]$;

(ii) $t[s] = [t^2 s]$.

Proof. By Lemma 2.7.51, we have that $s[t] = [t][s] = [tq[s]]$. Since $[s] = q^{-1}(s)$ by definition, it follows that $s[t] = [st]$. On the other hand, $t[s] = tq^{-1}(s) = q^{-1}(t^2 s) = [t^2 s]$. □

Now choose fixed arbitrary elements $\xi \in W \setminus Y$ and $d \in V \setminus R$.

Theorem 2.7.98. *There exists an element $e \in V$ such that $f(d,e) = 1$ and $f(d, e\xi) = 0$. Moreover, $f(d\xi, e\xi) = \hat{q}(\xi) \in L \setminus K^2$.*

Proof. We will first show the last statement. So let $a, b \in V$ be arbitrary elements such that $f(a,b) = 1$. Then, by Lemma 2.7.81(i) and Lemma 2.7.91(iv), we have that

$$f(a\xi, b\xi) = f(a, b\xi\xi) = f(a, q[\hat{q}(\xi)]b) = q[\hat{q}(\xi)]f(a,b) = q[\hat{q}(\xi)] = \hat{q}(\xi)$$

(note that $[s] = q^{-1}(s)$ for all $s \in L$ by definition). Let $\alpha := \hat{q}(\xi)$. Suppose that $\alpha \in K^2$, say $\alpha = t^2$ for some $t \in K$. Then $q(t\epsilon) = t^2 = \alpha = \hat{q}(\xi) = q(\epsilon\xi)$, hence $\epsilon[t] = t\epsilon = \epsilon\xi$. Since $[t] \in Y$, this implies that $\epsilon(\xi + [t]) = \epsilon\xi + \epsilon[t] = 0$, and hence $\xi = [t] \in Y$, which contradicts the choice of ξ. Hence $\alpha \notin K^2$.

Since $d \notin R = \mathrm{Rad}(F)$, there exist an elements $u \in V$ such that $F(d,u) \neq 0$. Let $v := f(d,u)^{-1}u$, then

$$\begin{aligned} f(d,v) &= f(d, f(d,u)^{-1}u) \\ &= f(d,u)^{-1} f(d,u) \\ &= 1 \,. \end{aligned}$$

In particular, $f(d\xi, v\xi) = \hat{q}(\xi) = \alpha$. Since $\alpha \notin K^2$, we also have that $\alpha^{-1} f(d, v\xi)^2 \neq 1$. Now let

$$e := \big(1 + \alpha^{-1} f(d, v\xi)^2\big)^{-1} \big(v + \alpha^{-1} f(d, v\xi) v\xi\big) \,.$$

Then, by Lemma 2.7.81(i),

$$\begin{aligned} f(d,e) &= \big(1 + \alpha^{-1} f(d,v\xi)^2\big)^{-1} f(d, v + \alpha^{-1} f(d,v\xi) v\xi) \\ &= \big(1 + \alpha^{-1} f(d,v\xi)^2\big)^{-1} \big(f(d,v) + \alpha^{-1} f(d,v\xi) f(d,v\xi)\big) \\ &= \big(1 + \alpha^{-1} f(d,v\xi)^2\big)^{-1} \big(1 + \alpha^{-1} f(d,v\xi)^2\big) \\ &= 1 \,, \end{aligned}$$

and

$$\begin{aligned} f(d,e\xi) &= \big(1 + \alpha^{-1} f(d,v\xi)^2\big)^{-1} f(d, v\xi + \alpha^{-1} f(d,v\xi) v\xi\xi) \\ &= \big(1 + \alpha^{-1} f(d,v\xi)^2\big)^{-1} \big(f(d,v\xi) + \alpha^{-1} f(d,v\xi) f(d,v\xi\xi)\big) \\ &= \big(1 + \alpha^{-1} f(d,v\xi)^2\big)^{-1} \big(f(d,v\xi) + \alpha^{-1} f(d,v\xi)\alpha\big) \\ &= 0 \,, \end{aligned}$$

which shows that e fulfills the required properties. $\square$

From now on, let $e \in V$ be as in Theorem 2.7.98, and let $\alpha := f(d\xi, e\xi) = \hat{q}(\xi)$. By Theorem 2.7.98, $\alpha \in L \setminus K^2$.

Theorem 2.7.99. *Let* $B := \langle d, e, d\xi, e\xi \rangle$. *Then* $\dim_K B = 4$ *and* $B \cap R = 0$.

Proof. Let $v = t_1 d + t_2 e + t_3 d\xi + t_4 e\xi$ with $t_1, t_2, t_3, t_4 \in K$ be an arbitrary element of B. Suppose that $v \in R = \mathrm{Rad}(F) = \mathrm{Rad}(f)$. Then $f(v, d) = f(v, e) = f(v, d\xi) = f(v, e\xi) = 0$. Observe that $f(d, d) = f(d, d\xi) = f(e, e) = f(e, e\xi) = f(d\xi, d\xi) = f(e\xi, e\xi) = 0$ by $(\mathbf{Q}_{11})$, and that $f(d, e\xi) = f(e, d\xi) = 0$ by Theorem 2.7.98 and Lemma 2.7.81(i). Moreover, we have that $f(d, e) = 1$ and $f(d\xi, e\xi) = \alpha \neq 0$. It now follows from $f(v, d) = 0$ that $t_2 = 0$, from $f(v, e) = 0$ that $t_1 = 0$, from $f(v, d\xi) = 0$ that $t_4 = 0$ and from $f(v, e\xi) = 0$ that $t_3 = 0$. Hence $v = 0$. This shows that $B \cap R = 0$.

Since $0 \in R$, the previous paragraph also shows that it follows from $v = 0$ that $t_1 = t_2 = t_3 = t_4 = 0$, hence d, e, $d\xi$ and $e\xi$ are linearly independent. It follows that $\dim_K B = 4$. $\square$

Theorem 2.7.100. $B^\perp = R$, *where* $B^\perp := \{v \in V \mid f(v, B) = 0\}$.

Proof. It is obvious that $R \subseteq B^\perp$. So let g be an arbitrary element of $B^\perp$. Then $f(g, d) = f(g, e) = f(g, d\xi) = f(g, e\xi) = 0$. If we substitute ξ for z, ξde for w and g for v in Lemma 2.7.96(i), then we get that

$$\xi de \cdot g\xi + \hat{q}(\xi)\xi deg = \hat{f}(\xi de, \xi g)\xi + \hat{f}(\xi de, \xi)\xi g\ .$$

Since $f(e, g\xi) = 0$ and $f(d, g\xi) = 0$, it follows from Lemma 2.7.89(i) that $\xi de \cdot g\xi = \xi d \cdot g\xi \cdot e = \xi \cdot g\xi \cdot d \cdot e$, and hence $\xi de \cdot g\xi = \hat{q}(\xi)\xi gde$ by Lemma 2.7.94(i). On the other hand, since $f(e, g) = 0$ and $f(d, g) = 0$, it follows from Lemma 2.7.89(i) that $\hat{q}(\xi)\xi deg = \hat{q}(\xi)\xi dge = \hat{q}(\xi)\xi gde$.

Hence $\xi de \cdot g\xi = \hat{q}(\xi)\xi deg$, and therefore $\hat{f}(\xi de, \xi g)\xi = \hat{f}(\xi de, \xi)\xi g$. By Lemma 2.7.81(i) and $(\mathbf{Q}_{11})$, we have that

$$\begin{aligned}
[\hat{f}(\xi de, \xi)] &= [\hat{f}(\xi d, \xi e)] \\
&= [\hat{q}(\xi d + \xi e)] + [\hat{q}(\xi d)] + [\hat{q}(\xi e)] \\
&= [\hat{q}(\xi(d + e))] + [\hat{q}(\xi d)] + [\hat{q}(\xi e)]
\end{aligned}$$

since $F(d\xi, e) = 0$. It follows from Lemma 2.7.95(i) that

$$\begin{aligned}
[\hat{f}(\xi de, \xi)] &= q(d + e)[\hat{q}(\xi)] + q(d)[\hat{q}(\xi)] + q(e)[\hat{q}(\xi)] \\
&= \big(q(d + e) + q(d) + q(e)\big)[\alpha] \\
&= f(d, e)[\alpha] \\
&= [\alpha]\ ,
\end{aligned}$$

and hence $\hat{f}(\xi de, \xi) = \alpha \neq 0$. It follows that $\xi g = \alpha^{-1}\hat{f}(\xi de, \xi g)\xi = \xi r$ with $r = [\alpha^{-1}\hat{f}(\xi de, \xi g)] \in R$. Since $\xi(g + r) = \xi g + \xi r = 0$ by ($\mathbf{Q}_{11}$), we conclude that $g = r \in R$, which completes the proof of this theorem. $\square$

Since $\dim_K B = 4$ is finite by Theorem 2.7.99, we have $V = B + B^{\perp}$. Since $B \cap R = 0$ by Theorem 2.7.99 and $B^{\perp} = R$ by Theorem 2.7.100, it follows that V has a decomposition $V = B \oplus R$. In particular, every complement of R in V has dimension 4 over K. By symmetry, it also follows that every complement of Y in W has dimension 4 over L.

Let $\beta := q(d)^{-1}$. Then $\beta \in K \setminus L$, since $\beta \in L$ would imply that $q(d) = \beta^{-1} = q[\beta^{-1}]$ and hence $d = [\beta^{-1}] \in [L] = R = \mathrm{Rad}(f)$, which contradicts the fact that $f(d,e) = 1$.

Theorem 2.7.101. *Let $A := \langle \xi, \xi ed^{-1}, \xi d^{-1}, \beta^2\xi e\rangle$. Then $W = A \oplus Y$.*

Proof. Let $w = s_1\xi + s_2\xi ed^{-1} + s_3\xi d^{-1} + s_4\beta^2\xi e$ with $s_1, s_2, s_3, s_4 \in L$ be an arbitrary element of A. Suppose that $w \in Y = \mathrm{Rad}(H) = \mathrm{Rad}(\hat{f})$. Observe that $\hat{q}[q(d)] = q(\epsilon[q(d)]) = q(q(d)\epsilon) = q(d)^2 = \beta^{-2}$ and hence, by Lemma 2.7.91(iii), $\hat{f}(\xi ed^{-1}, \xi) = \hat{f}(\xi e, \xi d^{-1}) = \hat{q}[q(d)]^{-1}\hat{f}(\xi e, \xi d) = \alpha\beta^2 \neq 0$.

By ($\mathbf{Q}_{12}$), $\hat{f}(\xi, \xi) = \hat{f}(\xi d^{-1}, \xi) = \hat{f}(\xi e, \xi) = 0$. It thus follows from $\hat{f}(w, \xi) = 0$ that $\hat{f}(s_2\xi ed^{-1}, \xi) = 0$ and hence $s_2 = 0$. We now have that $w = s_1\xi + s_3\xi d^{-1} + s_4\beta^2\xi e$.

Since $\hat{f}(\xi, \xi e) = \hat{f}(\xi e, \xi e) = 0$ and $\hat{f}(\xi d^{-1}, \xi e) = \alpha\beta^2 \neq 0$, it follows from $\hat{f}(w, \xi e) = 0$ that $s_3 = 0$, and hence $w = s_1\xi + s_4\beta^2\xi e$.

Since $\hat{f}(\xi, \xi d^{-1}) = 0$ and $\hat{f}(\xi e, \xi d^{-1}) = \alpha\beta^2 \neq 0$, it now follows from $\hat{f}(w, \xi d^{-1}) = 0$ that $s_4 = 0$. Hence $w = s_1\xi$.

Finally, it follows from $\hat{f}(w, \xi ed^{-1}) = 0$ that $\hat{f}(s_1\xi, \xi ed^{-1}) = s_1\alpha\beta^2 = 0$ and hence $s_1 = 0$.

So we have shown that $w \in Y$ implies $w = 0$, and at the same time, we have shown that ξ, ξed^{-1}, ξd^{-1} and $\beta^2\xi e$ are linearly independent. Hence $\dim_L A = 4$ and $A \cap Y = 0$, from which it follows that A is contained in a complement of Y in W. Since every complement of Y in W is 4-dimensional, this implies that A itself is a complement of Y, that is, $W = A \oplus Y$. $\square$

Let E be the splitting field of the polynomial $\phi(x) \equiv q(d)x^2 + x + q(e)$ over K.

Lemma 2.7.102. *E/K is a separable quadratic extension.*

Proof. Suppose that $t \in K$ would be a root of ϕ. Then

$$q(td + e) = q(td) + f(td, e) + q(e) = t^2q(d) + tf(d,e) + q(e) = \phi(t) ,$$

since $f(d,e) = 1$, hence $q(td + e) = 0$. Since q is anisotropic, this implies that $td + e = 0$, which contradicts the fact that d and e are linearly independent. Hence ϕ has no roots in K, so E/K is a quadratic extension. Since the coefficient of x of ϕ is non-zero, the two roots of ϕ are distinct, hence the extension is separable. □

Let $\omega \in E \setminus K$ be one of the roots of ϕ. Let $D := E^2L = L(\omega^2)$. Then D is the splitting field of the polynomial $\phi'(x) \equiv q(d)^2x^2 + x + q(e)^2$ over L. For both extensions E/K and D/L, we will denote the norm by N and the non-trivial element of the Galois group by $x \mapsto \overline{x}$.

We can consider E as a 2-dimensional vector space over K, and D as a 2-dimensional vector space over L. Let $B_0 := E \oplus E$, and let $A_0 := D \oplus D$. Then B_0 is a 4-dimensional vector space over K, and A_0 is a 4-dimensional vector space over L. We can identify B and A with B_0 and A_0, respectively, by the following relations.

$$\begin{aligned} t_1d + t_2e + t_3d\xi + t_4e\xi \quad &\longleftrightarrow \quad (t_1 + t_2\omega, t_3 + t_4\omega) \\ s_1\xi + s_2\xi ed^{-1} + s_3\xi d^{-1} + s_4\beta^2\xi e \quad &\longleftrightarrow \quad (s_1 + s_2\omega^2, s_3 + s_4\omega^2) \end{aligned}$$

Since $R = [L]$ and $Y = [K]$, we have actually identified V and W with $B_0 \oplus L$ and $A_0 \oplus K$, respectively:

$$\begin{aligned} t_1d + t_2e + t_3d\xi + t_4e\xi + [s] \quad &\longleftrightarrow \quad (t_1 + t_2\omega, t_3 + t_4\omega, s) \\ s_1\xi + s_2\xi ed^{-1} + s_3\xi d^{-1} + s_4\beta^2\xi e + [t] \quad &\longleftrightarrow \quad (s_1 + s_2\omega^2, s_3 + s_4\omega^2, t) \end{aligned}$$

For all $(b,s) \in B_0 \oplus L$ and all $(a,t) \in A_0 \oplus K$, we will denote the corresponding elements of V and W by $[b,s]$ and $[a,t]$, respectively.

We can now describe the quadratic forms q and $\hat{q}$ on $B_0 \oplus L$ and $A_0 \oplus K$, respectively, via this identification.

Theorem 2.7.103. *For all $u, v \in E$, $s \in L$, $x, y \in D$ and $t \in K$, we have that*

(i) $q[u,v,s] = \beta^{-1}(N(u) + \alpha N(v)) + s$;

(ii) $\hat{q}[x,y,t] = \alpha(N(x) + \beta^2 N(y)) + t^2$.

Proof. Let $u = t_1 + t_2\omega$ and $v = t_3 + t_4\omega$ be arbitrary elements of E, and let s be an arbitrary element of L. Then we have that

$$\begin{aligned} q[u,v,s] &= q[t_1 + t_2\omega, t_3 + t_4\omega, s] \\ &= q(t_1d + t_2e + t_3d\xi + t_4e\xi + [s]) \\ &= q(t_1d + t_2e) + q(t_3d\xi + t_4e\xi) + q[s] , \end{aligned}$$

since $f(t_1 d + t_2 e, t_3 d\xi + t_4 e\xi) = 0$ and $[s] \in \mathrm{Rad}(f)$. By Lemma 2.7.95(ii) and Lemma 2.7.97(i), $[q(v\xi)] = \alpha[q(v)] = [\alpha q(v)]$, and hence $q(v\xi) = \alpha q(v)$ for all $v \in V$. It follows that

$$\begin{aligned} q[u,v,s] &= q(t_1 d + t_2 e) + \alpha q(t_3 d + t_4 e) + q[s] \\ &= q(t_1 d) + f(t_1 d, t_2 e) + q(t_2 e) \\ &\qquad + \alpha(q(t_3 d) + f(t_3 d, t_4 e) + q(t_4 e)) + q[s] \\ &= t_1^2 q(d) + t_1 t_2 + t_2^2 q(e) + \alpha(t_3^2 q(d) + t_3 t_4 + t_4^2 q(e)) + s \\ &= q(d) N(t_1 + t_2\omega) + \alpha q(d) N(t_3 + t_4\omega) + s \\ &= \beta^{-1}(N(u) + \alpha N(v)) + s\ , \end{aligned}$$

which proves (i). Similarly, let $x = s_1 + s_2\omega^2$ and $y = s_3 + s_4\omega^2$ be arbitrary elements of D, and let t be an arbitrary element of K. Then we have that

$$\begin{aligned} \hat{q}[x,y,t] &= \hat{q}[s_1 + s_2\omega^2, s_3 + s_4\omega^2, t] \\ &= \hat{q}(s_1\xi + s_2\xi e d^{-1} + s_3\xi d^{-1} + s_4\beta^2\xi e + [t])\ . \end{aligned}$$

Note that $\hat{f}(\xi e d^{-1}, \xi d^{-1}) = \hat{f}(\xi e, \xi d^{-1} d^{-1}) = \hat{q}[q(d)]^{-1}\hat{f}(\xi e, \xi) = 0$ by Lemma 2.7.91(iii), hence $\hat{f}(s_1\xi + s_2\xi e d^{-1}, s_3\xi d^{-1} + s_4\beta^2\xi e) = 0$. Since $[t] \in \mathrm{Rad}(\hat{f})$, it thus follows that

$$\begin{aligned} \hat{q}[x,y,t] &= \hat{q}(s_1\xi + s_2\xi e d^{-1}) + \hat{q}(s_3\xi d^{-1} + s_4\beta^2\xi e) + \hat{q}[t] \\ &= \hat{q}(s_1\xi) + \hat{f}(s_1\xi, s_2\xi e d^{-1}) + \hat{q}(s_2\xi e d^{-1}) \\ &\qquad + \hat{q}(s_3\xi d^{-1}) + \hat{f}(s_3\xi d^{-1}, s_4\beta^2\xi e) + \hat{q}(s_4\beta^2\xi e) + \hat{q}[t] \\ &= s_1^2\hat{q}(\xi) + s_1 s_2\hat{f}(\xi, \xi e d^{-1}) + s_2^2\hat{q}(\xi e d^{-1}) \\ &\qquad + s_3^2\hat{q}(\xi d^{-1}) + s_3 s_4\beta^2\hat{f}(\xi, \xi e d^{-1}) + s_4^2\beta^4\hat{q}(\xi e) + \hat{q}[t]\ . \end{aligned}$$

By Lemma 2.7.95(i) and Lemma 2.7.97(ii),

$$[\hat{q}(wv)] = q(v)[\hat{q}(w)] = [q(v)^2\hat{q}(w)]\ ,$$

and hence $\hat{q}(wv) = q(v)^2\hat{q}(w)$ for all $v \in V$ and all $w \in W$. Remember that $\hat{f}(\xi, \xi e d^{-1}) = \alpha\beta^2$ and that $q(d^{-1}) = q(d)^{-1} = \beta$. Since $\hat{q}[t] = q(\epsilon[t]) = q(t\epsilon) = t^2$, it thus follows that

$$\begin{aligned} \hat{q}[x,y,t] &= s_1^2\alpha + s_1 s_2\alpha\beta^2 + s_2^2 q(e)^2 q(d)^{-2}\alpha \\ &\qquad + s_3^2 q(d)^{-2}\alpha + s_3 s_4\beta^2\alpha\beta^2 + s_4^2\beta^4 q(e)^2\alpha + t^2 \\ &= \alpha(s_1^2 + s_1 s_2 q(d)^{-2} + s_2^2 q(e)^2 q(d)^{-2} \\ &\qquad + \beta^2(s_3^2 + s_3 s_4 q(d)^{-2} + s_4^2 q(e)^2 q(d)^{-2})) + t^2 \\ &= \alpha(N(s_1 + s_2\omega^2) + \beta^2 N(s_3 + s_4\omega^2)) + t^2 \\ &= \alpha(N(x) + \beta^2 N(y)) + t^2\ , \end{aligned}$$

which proves (ii). $\square$

For all $a \in A_0$ and all $b \in B_0$, we let $q_1(b) := q[b,0]$ and $q_2(a) := \hat{q}[a,0]$. Denote the corresponding bilinear forms by f_1 and f_2, respectively. We now define maps $\tilde{\Upsilon}$, $\tilde{\nu}$, $\tilde{\Theta}$ and $\tilde{\psi}$ from $A_0 \times B_0$ to A_0, K, B_0 and L, respectively, by setting

$$[a,0][b,0] = [\tilde{\Upsilon}(a,b), \tilde{\nu}(a,b)] \,,$$
$$[b,0][a,0] = [\tilde{\Theta}(a,b), \tilde{\psi}(a,b)] \,,$$

for all $a \in A_0$ and all $b \in B_0$. We will show that these maps coincide with the maps Υ, ν, Θ and ψ defined on page 69.

Lemma 2.7.104. $\tilde{\Upsilon} \equiv \Upsilon$.

Proof. All the equivalences in the proof of this lemma are modulo Y. Let

$$\begin{aligned} a_1 &:= \xi \,, & b_1 &:= d \,, \\ a_2 &:= \xi e d^{-1} \,, & b_2 &:= e \,, \\ a_3 &:= \xi d^{-1} \,, & b_3 &:= d\xi \,, \\ a_4 &:= \beta^2 \xi e \,, & b_4 &:= e\xi \,, \end{aligned}$$

and let $a_{ij} := a_i b_j$ for all $i, j \in \{1,2,3,4\}$. We first observe that $\xi de + \xi ed \equiv f(d,e)^2 \xi \equiv \xi$ and that $\xi d^{-1} e + \xi e d^{-1} \equiv f(d^{-1}, e)^2 \xi \equiv \beta^2 \xi$ by Lemma 2.7.93(i). Then

$$\begin{aligned}
a_{11} &\equiv \xi \cdot d \equiv \beta^{-2} a_3 \,; \\
a_{12} &\equiv \xi \cdot e \equiv \beta^{-2} a_4 \,; \\
a_{13} &\equiv \xi \cdot d\xi \equiv \alpha \xi d \equiv \alpha \beta^{-2} a_3 \,; \\
a_{14} &\equiv \xi \cdot e\xi \equiv \alpha \xi e \equiv \alpha \beta^{-2} a_4 \,; \\
a_{21} &\equiv \xi e d^{-1} \cdot d \equiv \xi e \equiv \beta^{-2} a_4 \,; \\
a_{22} &\equiv \xi e d^{-1} \cdot e \equiv (\xi d^{-1} e + \beta^2 \xi) \cdot e \equiv q(e)^2 a_3 + a_4 \,; \\
a_{23} &\equiv \xi e d^{-1} \cdot d\xi \equiv \xi e \cdot d\xi \cdot d^{-1} \equiv \xi \cdot d\xi \cdot e d^{-1} \equiv \alpha \xi d e d^{-1} \\
&\equiv \alpha(\xi + \xi e d) d^{-1} \equiv \alpha a_3 + \alpha \beta^{-2} a_4 \,; \\
a_{24} &\equiv \xi e d^{-1} \cdot e\xi \equiv \xi e \cdot e\xi \cdot d^{-1} \equiv \xi \cdot e\xi \cdot e d^{-1} \equiv \alpha \xi e e d^{-1} \equiv \alpha q(e)^2 a_3; \\
a_{31} &\equiv \xi d^{-1} \cdot d \equiv a_1 \,; \\
a_{32} &\equiv \xi d^{-1} \cdot e \equiv \beta^2 \xi + \xi e d^{-1} \equiv \beta^2 a_1 + a_2 \,; \\
a_{33} &\equiv \xi d^{-1} \cdot d\xi \equiv \xi \cdot d\xi \cdot d^{-1} \equiv \alpha \xi d d^{-1} \equiv \alpha a_1 \,; \\
a_{34} &\equiv \xi d^{-1} \cdot e\xi \equiv \xi \cdot e\xi \cdot d^{-1} \equiv \alpha \xi e d^{-1} \equiv \alpha a_2 \,;
\end{aligned}$$

$$\begin{aligned}
a_{41} &\equiv \beta^2\xi e \cdot d \equiv q(d)^2\beta^2\xi e d^{-1} \equiv a_2\,;\\
a_{42} &\equiv \beta^2\xi e \cdot e \equiv \beta^2 q(e)^2 a_1\,;\\
a_{43} &\equiv \beta^2\xi e \cdot d\xi \equiv \beta^2\xi \cdot d\xi \cdot e \equiv \beta^2\alpha\xi de \equiv \alpha\beta^2(\xi + \xi ed) \equiv \alpha\beta^2 a_1 + \alpha a_2\,;\\
a_{44} &\equiv \beta^2\xi e \cdot e\xi \equiv \beta^2\xi \cdot e\xi \cdot e \equiv \beta^2\alpha\xi ee \equiv \alpha\beta^2 q(e)^2 a_1\,.
\end{aligned}$$

Hence

$$\begin{aligned}
\tilde\Upsilon\big((1,0),(1,0)\big) &= (0,\beta^{-2})\,; & \tilde\Upsilon\big((0,1),(1,0)\big) &= (1,0)\,;\\
\tilde\Upsilon\big((1,0),(\omega,0)\big) &= (0,\beta^{-2}\omega^2)\,; & \tilde\Upsilon\big((0,1),(\omega,0)\big) &= (\beta^2+\omega^2,0)\,;\\
\tilde\Upsilon\big((1,0),(0,1)\big) &= (0,\alpha\beta^{-2})\,; & \tilde\Upsilon\big((0,1),(0,1)\big) &= (\alpha,0)\,;\\
\tilde\Upsilon\big((1,0),(0,\omega)\big) &= (0,\alpha\beta^{-2}\omega^2)\,; & \tilde\Upsilon\big((0,1),(0,\omega)\big) &= (\alpha\omega^2,0)\,;
\end{aligned}$$

$$\begin{aligned}
\tilde\Upsilon\big((\omega^2,0),(1,0)\big) &= (0,\beta^{-2}\omega^2)\,; & \tilde\Upsilon\big((0,\omega^2),(1,0)\big) &= (\omega^2,0)\,;\\
\tilde\Upsilon\big((\omega^2,0),(\omega,0)\big) &= (0,q(e)^2+\omega^2)\,; & \tilde\Upsilon\big((0,\omega^2),(\omega,0)\big) &= (\beta^2 q(e)^2,0)\,;\\
\tilde\Upsilon\big((\omega^2,0),(0,1)\big) &= (0,\alpha+\alpha\beta^{-2}\omega^2)\,; & \tilde\Upsilon\big((0,\omega^2),(0,1)\big) &= (\alpha\beta^2+\alpha\omega^2,0)\,;\\
\tilde\Upsilon\big((\omega^2,0),(0,\omega)\big) &= (0,\alpha q(e)^2)\,; & \tilde\Upsilon\big((0,\omega^2),(0,\omega)\big) &= (\alpha\beta^2 q(e)^2,0)\,.
\end{aligned}$$

Since $\omega^2 = \beta\omega + \beta q(e)^2$ and $\overline{\omega} = \omega + \beta$, it is now straightforward to check that $\tilde\Upsilon$ coincides with the map Υ defined on page 69 on the set

$$\{(1,0),(\omega^2,0),(0,1),(0,\omega^2)\} \times \{(1,0),(\omega,0),(0,1),(0,\omega)\}\ .$$

By ($\mathbf{Q}_3$) and ($\mathbf{Q}_{11}$), the map $\tilde\Upsilon$ is additive in both variables. Since $(sw)v = s(wv)$ for all $s \in L$, $v \in V$ and $w \in W$, it follows that $\tilde\Upsilon(sa,b) = s\tilde\Upsilon(a,b)$ for all $s \in L$, $a \in A_0$ and $b \in B_0$. By Lemma 2.7.91(i), we have that $w(tv) = \hat q[t]wv = t^2 wv$ for all $t \in K$, $v \in V$ and $w \in W$, and hence $\tilde\Upsilon(a,tb) = t^2\tilde\Upsilon(a,b)$ for all $t \in K$, $a \in A_0$ and $b \in B_0$. Since the same properties hold for Υ, and since $A_0 = \langle (1,0),(\omega^2,0),(0,1),(0,\omega^2)\rangle$ and $B_0 = \langle (1,0),(\omega,0),(0,1),(0,\omega)\rangle$, it thus follows that $\tilde\Upsilon \equiv \Upsilon$. □

Lemma 2.7.105. *For all $a \in A_0$ and all $b, b' \in B_0$, we have that*

(i) $q_2(\tilde\Upsilon(a,b)) = q_1(b)^2 q_2(a) + \tilde v(a,b)^2$;

(ii) $\tilde v(a,b+b') = \tilde v(a,b) + \tilde v(a,b') + f_1(\tilde\Theta(a,b),b')$;

(iii) $q_1(\tilde\Theta(a,b)) = q_2(a)q_1(b) + \tilde\psi(a,b)$.

Proof. By Lemma 2.7.95(i) and Lemma 2.7.97(ii), $[\hat q(wv)] = q(v)[\hat q(w)] = [q(v)^2\hat q(w)]$, and hence $\hat q(wv) = q(v)^2\hat q(w)$ for all $v \in V$ and all $w \in W$. If we choose $v = [b,0]$ and $w = [a,0]$, then we get that

$$\hat q[\tilde\Upsilon(a,b),\tilde v(a,b)] = q[b,0]^2\hat q[a,0]\ .$$

Hence $q_2(\tilde{\Upsilon}(a,b)) + \tilde{\nu}(a,b)^2 = q_1(b)^2 q_2(a)$, which proves (i).

Similarly, it follows from Lemma 2.7.95(ii) and Lemma 2.7.97(i) that $q(vw) = \hat{q}(w)q(v)$ for all $v \in V$ and all $w \in W$. It follows that

$$q(\tilde{\Theta}(a,b), \tilde{\psi}(a,b)) = \hat{q}[a,0]q[b,0] \; .$$

Hence $q_1(\tilde{\Theta}(a,b)) + \tilde{\psi}(a,b) = q_2(a)q_1(b)$, which proves (iii).

Finally, it follows from ($\mathbf{Q}_{11}$) that

$$[a,0]\cdot[b+b',0] = [a,0]\cdot[b,0] + [a,0]\cdot[b',0] + F([b,0]\cdot[a,0],[b',0]) \; .$$

Projecting this identity on $Y = [0,K]$ yields

$$\begin{aligned} \nu(a,b+b') &= \nu(a,b) + \nu(a,b') + f([\tilde{\Theta}(a,b),\tilde{\psi}(a,b)],[b',0]) \\ &= \nu(a,b) + \nu(a,b') + f_1(\tilde{\Theta}(a,b),b') \; , \end{aligned}$$

which proves (ii). □

Theorem 2.7.106. *$\tilde{\Upsilon} \equiv \Upsilon$, $\tilde{\nu} \equiv \nu$, $\tilde{\Theta} \equiv \Theta$ and $\tilde{\psi} \equiv \psi$.*

Proof. We have already shown in Lemma 2.7.104 that $\tilde{\Upsilon} \equiv \Upsilon$. It then follows from Lemma 2.7.105(i) and Theorem 2.6.10(xii) that $\tilde{\nu} \equiv \nu$. Hence, by Lemma 2.7.105(ii) and Theorem 2.6.10(i), we have that

$$f_1(\tilde{\Theta}(a,b) - \Theta(a,b), b') = 0$$

for all $a \in A_0$ and all $b, b' \in B_0$, from which it follows that

$$\tilde{\Theta}(a,b) - \Theta(a,b) \in \mathrm{Rad}(f_1)$$

for all $a \in A_0$ and all $b \in B_0$. Since $B \cap \mathrm{Rad}(f) = B \cap R = 0$ by Theorem 2.7.99, we have that $B_0 \cap \mathrm{Rad}(f_1) = 0$ as well, and hence $\tilde{\Theta} \equiv \Theta$. It then follows from Lemma 2.7.105(iii) and Theorem 2.6.10(xi) that $\tilde{\psi} \equiv \psi$. □

Theorem 2.7.107. *$(V, W, \tau_V, \tau_W, \epsilon, \delta) \cong \Omega_F(K, V_0, q)$.*

Proof. First of all, observe that q is indeed a quadratic form of type F_4, since its regular component q_1 has a norm splitting

$$q_1(u,v) = \beta^{-1}N(u) + \beta^{-1}\alpha N(v) \; ,$$

and the product of its coefficients is $\beta^{-1}\cdot\beta^{-1}\alpha = \beta^{-2}\alpha$, which is an element of L.

Let ϕ be the isomorphism from $[V_0] = [B_0 \oplus L]$ to V which maps $[b,s]$ to $[b,s]$ for all $b \in B_0$ and all $s \in L$, and let ψ be the isomorphism from

$[W_0] = [A_0 \oplus K]$ to W which maps $[a,t]$ to $[a,t]$ for all $a \in A_0$ and all $t \in K$. Then $\phi([0,1]) = [0,1] = [1]_V = \epsilon$ and $\psi([0,1]) = [0,1] = [1]_W = \delta$.

Since ϕ and ψ are identity maps, it now only remains to show that it follows from the relations

$$\begin{aligned} [a,0][b,0] &= [\tilde{\Upsilon}(a,b), \tilde{\nu}(a,b)] \ , \\ [b,0][a,0] &= [\tilde{\Theta}(a,b), \tilde{\psi}(a,b)] \ , \end{aligned}$$

for all $a \in A_0$ and all $b \in B_0$ that

$$\begin{aligned} [a,t][b,s] &= [\tilde{\Upsilon}(a,b) + sa, \tilde{\nu}(a,b) + q[b,s]t] \ , \\ [b,s][a,t] &= [\tilde{\Theta}(a,b) + tb, \tilde{\psi}(a,b) + \hat{q}[a,t]s] \ , \end{aligned}$$

for all $a \in A_0$ and all $b \in B_0$. We will only show the first identity, the second one being completely similar. Since $[0,s] \in \mathrm{Rad}(F)$, it follows from $(\mathbf{Q}_{11})$ and Lemma 2.7.51 that

$$\begin{aligned} [a,t][b,s] &= [a,0][b,s] + [0,t][b,s] \\ &= [a,0][b,0] + [a,0][0,s] + [0,t][b,s] \\ &= [\tilde{\Upsilon}(a,b), \tilde{\nu}(a,b)] + [sa,0] + [0, tq[b,s]] \\ &= [\tilde{\Upsilon}(a,b) + sa, \tilde{\nu}(a,b) + q[b,s]t] \ . \end{aligned}$$

Since ϕ and ψ are identity maps, it is now obvious that $\phi([b,s][a,t]) = \phi([b,s])\psi([a,t])$ and $\psi([a,t][b,s]) = \psi([a,t])\phi([b,s])$ for all $(a,t) \in W_0$ and all $(b,s) \in V_0$; hence (ϕ,ψ) is an isomorphism from $\Omega_F(K, V_0, q)$ to $(V, W, \tau_V, \tau_W, \epsilon, \delta)$. □

2.7.6 Quadrangular Systems of Pseudo-quadratic Form Type, Part II

In this section, we continue to assume that $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ is a wide quadrangular system which is the extension of a quadrangular system Λ of quadratic form type, i.e. $\Lambda = (V, \mathrm{Rad}(H), \tau_V, \tau_W, \epsilon, \delta) \cong \Omega_Q(K, V_0, q)$, where τ_V and τ_W are as in Remark 2.7.3.

Our goal in this and the next section is to classify these quadrangular systems if $\mathrm{Rad}(F) = 0$. So assume that $\mathrm{Rad}(F) = 0$. We continue to identify V and V_0 if there is no danger of confusion.

Lemma 2.7.108. *For all $v \in V$, all $w \in W$ and all $t \in K$, we have that $(tv)w = t(vw)$. It follows that the notation tvw is unambiguous.*

Proof. If we substitute $[t]$ for z, $\overline{v}$ for v and $\boxminus w$ for w in ($\mathbf{Q}_{23}$), then we get, since $\Pi_w([t]) = [t]$, that

$$t\overline{v} = -\overline{\overline{(t \cdot \overline{v}w)}\kappa(\boxminus w)}\ ,$$

and hence, by Lemma 2.7.50(ii), that

$$\overline{tv} = -\overline{(t \cdot vw)\kappa(\boxminus w)}\ .$$

It follows that

$$tv \cdot w = -(t \cdot vw)\kappa(\boxminus w) \cdot w\ ,$$

and hence, by ($\mathbf{Q}_{18}$), that $tv \cdot w = t \cdot vw$, which is what we had to show. □

Definition 2.7.109. If $\mathrm{char}(K) \neq 2$, let $\zeta := \epsilon/2$. If $\mathrm{char}(K) = 2$, define $S_1 := \{v \in V \mid F(\epsilon, v) \neq 0\}$ and $S_2 := \{\epsilon w \mid w \in W\}$. If $S_1 \cap S_2 \neq \emptyset$, choose a fixed element $z \in S_1 \cap S_2$; if $S_1 \cap S_2 = \emptyset$, choose a fixed element $z \in S_1$. Observe that $S_1 \neq \emptyset$ since $\epsilon \notin \mathrm{Rad}(F)$. In both cases, we let $\zeta := f(\epsilon, z)^{-1}z$.

It follows that $f(\epsilon, \zeta) = 1$, independent of the characteristic.

Remark 2.7.110. This somewhat strange definition will become clear in section 2.7.7.

Definition 2.7.111. An element $w \in W$ is called *ζ-orthogonal* if and only if $f(\zeta, \epsilon w) = 0$.

Lemma 2.7.112. *Each coset of Y in W contains a unique ζ-orthogonal element.*

Proof. Consider an arbitrary coset $w \boxplus Y$ of Y in W (where $w \in W$). An arbitrary element of this coset, say $w \boxplus [t]$ with $t \in K$, is ζ-orthogonal if and only if $f(\zeta, \epsilon(w \boxplus [t])) = 0$. By ($\mathbf{Q}_{12}$), $f(\zeta, \epsilon(w \boxplus [t])) = f(\zeta, \epsilon w + \epsilon[t]) = f(\zeta, \epsilon w) + f(\zeta, t\epsilon) = f(\zeta, \epsilon w) + t$, hence $w \boxplus [t]$ is ζ-orthogonal if and only if $t = -f(\zeta, \epsilon w)$. □

Since $Y = \mathrm{Rad}(H)$ is a normal subgroup of W, we can define the quotient group $X := W/Y$. Since $[W, W]_\boxplus \leq Y$, the group X is abelian. We will use the additive notations $+$ and $-$ for X.

Definition 2.7.113. We now define a map $\rho : X \to W$ as follows. For each element $w \boxplus Y \in X$, we define $\rho(w \boxplus Y)$ to be the unique element $w \boxplus y \in w \boxplus Y \subseteq W$ which is ζ-orthogonal; see Lemma 2.7.112. Moreover, for all $x \in X$ and all $t \in K$, we let $(x, t) := \rho(x) \boxplus [t] \in W$. Note that $\rho(x) \in x$ for all $x \in X$, and hence $(x, t) \in x$ for all $x \in X$ and all $t \in K$ as well.

Lemma 2.7.114. *For all $w \in W$, there exist unique elements $x \in X$ and $t \in K$ such that $w = (x, t)$.*

Proof. Let $w \in W$ be arbitrary. Let $x := w \boxplus Y \in X$, and let

$$y := \boxminus\rho(x) \boxplus w \in \boxminus(w \boxplus Y) \boxplus w = \boxminus Y \boxminus w \boxplus w = Y \ .$$

Hence $y = [t]$ for some $t \in K$, and we thus have that $(x, t) = \rho(x) \boxplus [t] = \rho(x) \boxplus y = w$.

Now suppose that $(x_1, t_1) = (x_2, t_2)$ for some $x_1, x_2 \in X$ and some $t_1, t_2 \in K$. Since $(x_1, t_1) \in x_1$ and $(x_2, t_2) \in x_2$, it follows that the cosets x_1 and x_2 have an element in common, and hence they are equal, i.e. $x_1 = x_2$. It then follows from $(x_1, t_1) = (x_2, t_2)$ that $t_1 = t_2$ as well. $\square$

Definition 2.7.115. We define a map $G : X \times X \to W$ by setting

$$G(a, b) := \boxminus\rho(a + b) \boxplus \rho(a) \boxplus \rho(b)$$

for all $a, b \in X$. Note that a and b are cosets of Y in W. It follows that $G(a, b) \in -(a + b) + a + b = Y$. Hence we can define a map $g : X \times X \to K$ by setting $G(a, b) = [g(a, b)]$ for all $a, b \in X$.

Lemma 2.7.116. $(a, t) \boxplus (b, s) = (a + b, t + s + g(a, b))$ *for all $a, b \in X$ and all $t, s \in K$.*

Proof. Since $Y \subseteq Z(W)$ by $(\mathbf{Q}_8)$, we have that

$$\begin{aligned}
(a, t) \boxplus (b, s) &= \rho(a) \boxplus [t] \boxplus \rho(b) \boxplus [s] \\
&= \rho(a) \boxplus \rho(b) \boxplus [t + s] \\
&= \rho(a + b) \boxplus [g(a, b)] \boxplus [t + s] \\
&= (a + b, g(a, b) + t + s) \ ,
\end{aligned}$$

which is what we had to show. $\square$

Definition 2.7.117. We define a map θ from $X \times V$ to V, a map π from X to V and a map h from $X \times X$ to V by setting

$$\begin{aligned}
\theta(a, v) &:= v \cdot (a, 0) \ , \\
\pi(a) &:= \theta(a, \epsilon) = \epsilon \cdot (a, 0) \ , \\
h(a, b) &:= H\big((a, 0), (b, 0)\big) \ ,
\end{aligned}$$

for all $a, b \in X$ and all $v \in V$.

By definition, $(a, 0)$ is ζ-orthogonal for all $a \in X$, hence $f(\pi(a), \zeta) = f(\epsilon \cdot (a, 0), \zeta) = 0$ for all $a \in X$. Furthermore, it follows from ($\mathbf{Q}_{12}$) that $v \cdot (a, t) = v \cdot (a, 0) + v \cdot [t] = \theta(a, v) + tv$ for all $v \in V$, all $a \in X$ and all $t \in K$.

Definition 2.7.118. We define a map $(a, v) \mapsto av$ from $X \times V$ to X and a map φ from $X \times V$ to K by the relation

$$(a, 0) \cdot v = (av, \varphi(a, v)) \ .$$

Since we did not define a multiplication yet between elements of X and elements of V, this will not cause confusion.

Note that it follows from $(a, 0) \cdot \epsilon = (a, 0)$ that $a\epsilon = a$ and $\varphi(a, \epsilon) = 0$ for all $a \in X$. Furthermore, we have that $(a, t) \cdot v = (av, tq(v) + \varphi(a, v))$ by Lemma 2.7.51, and that $H\big((a, t), (b, s)\big) = h(a, b)$, for all $a, b \in X$, all $v \in V$ and all $t, s \in K$.

Lemma 2.7.119. *For all $a \in X$, we have that*

$$g(a, -a) = g(-a, a) = f(\pi(a), \epsilon) = f(\pi(-a), \epsilon) \ .$$

Proof. Let $w := (a, 0) \in W$. Then w is ζ-orthogonal. By 2.2.13(i), we have that $w(-\epsilon) = [f(\epsilon w, \epsilon)] \boxminus w$. By ($\mathbf{Q}_6$), $f(\epsilon \cdot w(-\epsilon), \zeta) = f(\epsilon w, \zeta) = 0$, and hence $w(-\epsilon)$ is ζ-orthogonal as well. It follows that $w(-\epsilon) = (b, 0)$ for some $b \in X$. Since $[f(\epsilon w, \epsilon)] \in Y$, we now have that $b = w(-\epsilon) \boxplus Y = [f(\epsilon w, \epsilon)] \boxminus w \boxplus Y = \boxminus w \boxplus Y = -a$. It follows by Lemma 2.7.116 that

$$\begin{aligned}
[f(\pi(a), \epsilon)] &= [f(\epsilon w, \epsilon)] \\
&= w(-\epsilon) \boxplus w \\
&= (-a, 0) \boxplus (a, 0) \\
&= (-a + a, 0 + 0 + g(-a, a)) \\
&= [g(-a, a)] \ .
\end{aligned}$$

Hence $f(\pi(a), \epsilon) = g(-a, a)$, and since it follows from ($\mathbf{Q}_6$) that

$$\begin{aligned}
f(\pi(-a), \epsilon) &= f(\epsilon \cdot (-a, 0), \epsilon) \\
&= f(\epsilon \cdot (a, 0)(-\epsilon), \epsilon) \\
&= f(\epsilon \cdot (a, 0), \epsilon) \\
&= f(\pi(a), \epsilon) \ ,
\end{aligned}$$

we conclude that $g(a, -a) = g(-a, a) = f(\pi(a), \epsilon) = f(\pi(-a), \epsilon)$. $\square$

Definition 2.7.120. We define a map $(t, a) \mapsto ta$ from $K \times X$ to X by setting $ta := a \cdot t\epsilon$ for all $t \in K$ and all $a \in X$. We will prove later on (see Theorem 2.7.123) that this makes X into a vector space over K.

Lemma 2.7.121. *For all $a \in X$ and all $t \in K$, we have that $\varphi(a, t\epsilon) = 0$. Moreover, for all $a \in X$, all $v \in V$ and all $t \in K$, we have that*

(i) $ta \cdot v = a \cdot tv = t \cdot av$;

(ii) $\varphi(ta, v) = \varphi(a, tv) = t^2\varphi(a, v)$.

Proof. Let $w := (a, 0) \in W$ and let $y := [t] \in Y$. Since $\Pi_{\boxminus y}(z) = z$ for all $z \in W$, it follows from ($\mathbf{Q}_{26}$) that $w \cdot \epsilon y \cdot v = w \cdot vy$, for all $v \in V$. It thus follows from ($\mathbf{Q}_{11}$) that

$$\begin{aligned} F(\epsilon \cdot w(\epsilon y), \zeta) &= \boxminus w \cdot \epsilon y \cdot \epsilon \boxminus w \cdot \epsilon y \cdot \zeta \boxplus w \cdot \epsilon y \cdot (\zeta + \epsilon) \\ &= \boxminus w \cdot \epsilon y \boxminus w \cdot \zeta y \boxplus w \cdot (\zeta + \epsilon) y \\ &= F(\epsilon y \cdot w, \zeta y) \\ &= F(t\epsilon w, t\zeta) \\ &= [t^2 f(\epsilon w, \zeta)] \\ &= 0 , \end{aligned}$$

since w is ζ-orthogonal. It follows that $w \cdot \epsilon y$ is ζ-orthogonal as well. Since

$$w \cdot \epsilon y = w \cdot t\epsilon = (a, 0) \cdot t\epsilon = (a \cdot t\epsilon, \varphi(a, t\epsilon)) ,$$

it follows that $\varphi(a, t\epsilon) = 0$.

It now follows from $w \cdot \epsilon y \cdot v = w \cdot vy$ that $(a \cdot t\epsilon, 0) \cdot v = (a, 0) \cdot tv$ for all $v \in V$, and hence

$$(ta \cdot v, \varphi(ta, v)) = (ta, 0) \cdot v = (a \cdot t\epsilon, 0) \cdot v = (a, 0) \cdot tv = (a \cdot tv, \varphi(a, tv))$$

for all $v \in V$. This implies that $ta \cdot v = a \cdot tv$ and $\varphi(ta, v) = \varphi(a, tv)$.

Now observe that $\pi_{t\epsilon}(c) = \pi_\epsilon(c) = -\bar{c}$, for all $c \in V$. If we substitute $t\epsilon$ for v and v for c in ($\mathbf{Q}_{24}$), we thus get that $wv = w \cdot t\epsilon \cdot v \cdot (t\epsilon)^{-1}$, and hence

$$(a, 0) \cdot v \cdot t\epsilon = (a, 0) \cdot t\epsilon \cdot v .$$

It follows that

$$(av, \varphi(a, v)) \cdot t\epsilon = (ta, 0) \cdot v ,$$

and finally, since $q(t\epsilon) = t^2$, that

$$(t \cdot av, t^2\varphi(a, v)) = (ta \cdot v, \varphi(ta, v)) ,$$

and we are done. $\square$

Lemma 2.7.122. *The map $(a, v) \mapsto av$ is additive in both variables. Moreover, the following hold for all $a, b \in X$ and all $u, v \in V$:*

(i) $\varphi(a+b, v) + g(a, b)q(v) = \varphi(a, v) + \varphi(b, v) + g(av, bv)$;

(ii) $\varphi(a, u+v) = \varphi(a, u) + \varphi(a, v) + g(av, au) + f(\theta(a, u), v)$.

Proof. It follows from ($\mathbf{Q}_3$) that

$$\big((a,0) \boxplus (b,0)\big)v = (a,0) \cdot v \boxplus (b,0) \cdot v ,$$

and hence, by Lemma 2.7.116, that

$$(a+b, g(a,b)) \cdot v = (av, \varphi(a,v)) \boxplus (bv, \varphi(b,v)) ,$$

from which it follows that

$$\begin{aligned}&\big((a+b)v, g(a,b)q(v) + \varphi(a+b,v)\big)\\ &\qquad\qquad = \big(av+bv, \varphi(a,v) + \varphi(b,v) + g(av,bv)\big) .\end{aligned}$$

So we have shown that $(a+b)v = av + bv$ and that (i) holds.

On the other hand, it follows from ($\mathbf{Q}_{11}$) that

$$(a,0) \cdot (u+v) = (a,0) \cdot (v+u) = (a,0) \cdot v \boxplus (a,0) \cdot u \boxplus F(u \cdot (a,0), v) ,$$

and hence

$$\begin{aligned}&\big(a(u+v), \varphi(a, u+v)\big)\\ &\qquad = (av, \varphi(a,v)) \boxplus (au, \varphi(a,u)) \boxplus [f(\theta(a,u), v)]\\ &\qquad = \big(av + au, \varphi(a,v) + \varphi(a,u) + g(av,au) + f(\theta(a,u),v)\big) .\end{aligned}$$

So we have shown that $a(u+v) = au + av$ and that (ii) holds. □

Theorem 2.7.123. *X_0 is a vector space over K, with the scalar multiplication given by the map $(t, a) \mapsto ta = a \cdot t\epsilon$.*

Proof. First of all, we have that $1a = a \cdot \epsilon = a$ for all $a \in X$. By Lemma 2.7.122, the two distributivity laws hold, since

$$t(a+b) = (a+b) \cdot t\epsilon = a \cdot t\epsilon + b \cdot t\epsilon = ta + tb$$

for all $t \in K$ and all $a, b \in X$, and

$$(s+t)a = a \cdot (s+t)\epsilon = a \cdot (s\epsilon + t\epsilon) = a \cdot s\epsilon + a \cdot t\epsilon = sa + ta$$

for all $s, t \in K$ and all $a \in X$. Finally, it follows from Lemma 2.7.121(i) that

$$st \cdot a = ts \cdot a = a \cdot (ts)\epsilon = a \cdot t(s\epsilon) = ta \cdot s\epsilon = s \cdot ta$$

for all $s, t \in K$ and all $a \in X$. □

Lemma 2.7.124. *For all $a, b \in X$, all $u, v \in V$ and all $t \in K$, we have that*

(i) $\theta(ta, v) = t^2\theta(a, v)$;

(ii) $\theta(a, tv) = t\theta(a, v)$;

(iii) $\theta(a + b, v) + g(a, b)v = \theta(a, v) + \theta(b, v) + h(b, av)$;

(iv) $\theta(a, u + v) = \theta(a, u) + \theta(a, v)$.

Proof. Let $w := (a, 0) \in W$. Note that $\pi_{t\epsilon}(c) = \pi_\epsilon(c) = -\overline{c}$, for all $c \in V$. It thus follows by substituting $t\epsilon$ for v and v for c in $(\mathbf{Q}_{25})$ that $v \cdot \delta(t\epsilon) \cdot w = v \cdot w(t\epsilon)$. Hence

$$\begin{aligned}\theta(ta, v) &= v \cdot (ta, 0) = v \cdot w(t\epsilon) = v \cdot \delta(t\epsilon) \cdot w \\ &= v \cdot [q(t\epsilon)] \cdot w = t^2 vw = t^2 v \cdot (a, 0) = t^2\theta(a, v) \ ,\end{aligned}$$

which proves (i). Since $t \cdot vw = tv \cdot w$, we have that $t\theta(a, v) = \theta(a, tv)$, which proves (ii).

It follows from $(\mathbf{Q}_{12})$ that

$$\begin{aligned}\theta(a + b, v) + g(a, b)v &= v \cdot (a + b, g(a, b)) \\ &= v \cdot \big((a, 0) \boxplus (b, 0)\big) \\ &= v \cdot (a, 0) + v \cdot (b, 0) + H\big((b, 0), (a, 0) \cdot v\big) \\ &= \theta(a, v) + \theta(b, v) + H\big((b, 0), (av, \varphi(a, v))\big) \\ &= \theta(a, v) + \theta(b, v) + h(b, av) \ ,\end{aligned}$$

which shows (iii). Finally, it follows from $(\mathbf{Q}_4)$ that

$$\begin{aligned}\theta(a, u + v) &= (u + v) \cdot (a, 0) \\ &= u \cdot (a, 0) + v \cdot (a, 0) \\ &= \theta(a, u) + \theta(a, v) \ ,\end{aligned}$$

which proves (iv). $\square$

Lemma 2.7.125. *For all $a, b \in X$ and all $t \in K$, we have that $h(ta, b) = h(a, tb) = th(a, b)$.*

Proof. If we substitute $t\epsilon$ for v in Lemma 2.7.124(iii), then we get, by Lemma 2.7.124(ii), that

$$\begin{aligned}h(b, a \cdot t\epsilon) &= \theta(a + b, t\epsilon) + g(a, b)t\epsilon - \theta(a, t\epsilon) - \theta(b, t\epsilon) \\ &= t\theta(a + b, \epsilon) + tg(a, b)\epsilon - t\theta(a, \epsilon) - t\theta(b, \epsilon) \\ &= th(b, a) \ ,\end{aligned}$$

hence $h(b,ta) = th(b,a)$. It follows by ($\mathbf{Q}_{22}$) and Lemma 2.7.50(ii) that

$$h(ta,b) = -\overline{h(b,ta)} = -\overline{th(b,a)} = -t\overline{h(b,a)} = th(a,b)$$

as well, and we are done. □

Lemma 2.7.126. *For all $a,b \in X$, we have that*

$$f(h(a,b),\epsilon) = g(b,a) - g(a,b) \ .$$

Proof. If we set $v = \epsilon$, $w_1 = (b,0)$ and $w_2 = (a,0)$ in ($\mathbf{Q}_8$), then we get that

$$\boxminus(b,0) \boxminus (a,0) \boxplus (b,0) \boxplus (a,0) = [f(h(a,b),\epsilon)] \ .$$

Since

$$\begin{aligned}\boxminus(b,0) \boxminus (a,0) \boxplus (b,0) \boxplus (a,0) &= \boxminus\big((a,0) \boxplus (b,0)\big) \boxplus \big((b,0) \boxplus (a,0)\big) \\ &= \boxminus(a+b, g(a,b)) \boxplus (a+b, g(b,a)) \\ &= (0, -g(a,b) + g(b,a)) \ ,\end{aligned}$$

it follows that $f(h(a,b),\epsilon) = -g(a,b) + g(b,a)$. □

Lemma 2.7.127. *For all $a,b \in X$ and all $v \in V$, we have that*

$$f(h(a,b),v) = f(h(av,b),\epsilon) = f(h(a,b\overline{v}),\epsilon) \ .$$

Proof. It follows from ($\mathbf{Q}_8$) that

$$F(H(w_2,w_1),v) = [w_1, w_2 v]_{\boxplus} = F(H(w_2 v, w_1),\epsilon)$$

for all $w_1, w_2 \in W$. If we choose $w_2 = (a,0)$ and $w_1 = (b,0)$, then we get that $f(h(a,b),v) = f(h(av,b),\epsilon)$. It then follows from Lemma 2.7.54 that

$$\begin{aligned} f\big(h(a,b),v\big) &= f\big(\overline{h(a,b)},\overline{v}\big) = -f\big(h(b,a),\overline{v}\big) \\ &= -f\big(h(b\overline{v},a),\epsilon\big) = -f\big(\overline{h(b\overline{v},a)},\overline{\epsilon}\big) \\ &= f\big(h(a,b\overline{v}),\epsilon\big)\end{aligned}$$

as well. □

Lemma 2.7.128. *We have that $av\overline{v} = q(v)a$ and $au\overline{v} + av\overline{u} = f(u,v)a$ for all $a \in X$ and all $u,v \in V$.*

Proof. Let $w := (a, 0) \in W$. It then follows from ($\mathbf{Q}_{15}$) that $avv^{-1} = a$. Since $q(v)v^{-1} = \overline{v}$, it follows from Lemma 2.7.121(i) that $q(v)a = q(v)avv^{-1} = av \cdot q(v)v^{-1} = av\overline{v}$. It then follows that

$$\begin{aligned} f(u,v)a &= q(u+v)a - q(u)a - q(v)a \\ &= a(u+v)(\overline{u}+\overline{v}) - au\overline{u} - av\overline{v} \\ &= au\overline{v} + av\overline{u} \end{aligned}$$

as well. □

We have now come to a point which is very similar to Chapter 26 in [52]. For some of the remaining identities, we will thus simply refer to the appropriate place in [52]. Note that [52] uses δ where we use ζ.

Lemma 2.7.129. *For all $a, b \in X$, we have that $g(a,b) = f(h(b,a), \zeta)$.*

Proof. See [52, (26.20)]. □

Since h is bilinear over K, it follows from Lemma 2.7.129 that g is bilinear over K.

Lemma 2.7.130. *For all $a, b \in X$ and all $v \in V$, we have that*

$$h(a, bv) - h(b, av) = f(h(a,b), \epsilon)v\ .$$

Proof. See [52, (26.23)]. □

Lemma 2.7.131. *If $\mathrm{char}(K) \neq 2$, then $\varphi \equiv 0$, and for all $a \in X$ and all $v \in V$, we have that*

(i) $g(a,a) = 0$;

(ii) $\theta(a,v) = \frac{1}{2}h(a,av)$.

Proof. See [52, (26.24)]. □

Note that it follows from Lemma 2.7.131(i) and the fact that g is bilinear over K that g is skew-symmetric if $\mathrm{char}(K) \neq 2$.

Lemma 2.7.132. *If $\mathrm{char}(K) = 2$, then*

(i) $h(a,av) = g(a,a)v = f(\epsilon, \pi(a))v$;

(ii) $f(\theta(a,v),v) = g(av,av) = g(a,a)q(v) = f(\epsilon,\pi(a))q(v)$;

(iii) $f(\theta(a,u),v) = f(\theta(a,v),u) + f(\epsilon,\pi(a))f(u,v)$;

for all $a \in X$ and all $u, v \in V$.

Proof. See [52, (26.25)]. □

Lemma 2.7.133. *For all $a \in X$ and all $u, v \in V$, we have that*

(i) $f(\theta(a,v),v) = f(\epsilon,\pi(a))q(v)$;

(ii) $f(\theta(a,v),u) + f(\theta(a,u),v) = f(\epsilon,\pi(a))f(u,v)$.

Proof. See [52, (26.26)]. □

Lemma 2.7.134. *For all $a \in X$, all $u \in V$ and all $v \in V^*$, we have that*

$$\begin{aligned}\theta(av^{-1},u) + \varphi(a,v^{-1})u &= q(v)^{-1}\overline{\theta(a,\overline{u})} - f(u,v')\overline{\theta(a,v^{-1})} \\ &\quad - f(\overline{\theta(a,\overline{u})},v)q(v)^{-1}v' + f(\overline{\theta(a,v^{-1})},v)f(u,v')v' ,\end{aligned}$$

where $v' = \overline{v^{-1}} = q(v)^{-1}v$.

Proof. Let $w := (a,0) \in W$, and let $c := q(v)^{-1}\pi_v(u) \in V$. Since $\delta v = [q(v)]$, it follows by substituting u for c in $(\mathbf{Q}_{25})$ and by Lemma 2.7.52 that

$$\pi_v(\overline{u \cdot w\overline{v}}) = q(v)\pi_v(\overline{u}) \cdot w .$$

Note that $\pi_{v^{-1}}(\overline{v_2}) = \overline{\pi_v(v_2)}$ for all $v_2 \in V$ by Lemma 2.2.18(i). If we replace v by v^{-1}, then it follows that

$$\overline{\pi_v(u \cdot wv^{-1})} = q(v^{-1})\overline{\pi_v(u)} \cdot w ,$$

and hence, since $q(v^{-1}) = q(v)^{-1}$, that

$$u \cdot wv^{-1} = \pi_v(\overline{\overline{c} \cdot w}) ,$$

from which it follows that

$$u \cdot (av^{-1}, \varphi(a,v^{-1})) = \pi_v(\overline{\theta(a,\overline{c})}) ,$$

and therefore

$$\theta(av^{-1},u) + \varphi(a,v^{-1})u = \overline{\theta(a,\overline{c})} - f(v,\overline{\theta(a,\overline{c})})v' .$$

Since

$$\overline{c} = q(v)^{-1}\overline{\pi_v(u)} = q(v)^{-1}\overline{u} - q(v)^{-1}f(v,u)v^{-1} = q(v)^{-1}\overline{u} - f(u,v')v^{-1} ,$$

it follows by Lemma 2.7.124 that

$$\begin{aligned}\theta(av^{-1},u) + \varphi(a,v^{-1})u &= q(v)^{-1}\overline{\theta(a,\overline{u})} - f(u,v')\overline{\theta(a,v^{-1})} \\ &\quad - q(v)^{-1}f(v,\overline{\theta(a,\overline{u})})v' + f(v,\overline{\theta(a,v^{-1})})f(u,v')v' ,\end{aligned}$$

which is what we had to show. □

We now define

$$v^* := \begin{cases} 0 & \text{if } \operatorname{char}(K) \neq 2 \\ f(v,\zeta)\epsilon + f(v,\epsilon)\zeta + v & \text{if } \operatorname{char}(K) = 2 \end{cases},$$

for all $v \in V$.

Lemma 2.7.135. *If* $\operatorname{char}(K) = 2$*, then*

(i) $\varphi(a,v) = f(\theta(a,v^*),v)$
$\qquad = f(\pi(a),v)f(\zeta,v) + f(\theta(a,\zeta),v)f(\epsilon,v) + f(\epsilon,\pi(a))q(v)\,;$

(ii) *If* $f(\epsilon,v) = f(\zeta,v) = 0$*, then* $\pi(av) = \pi(a)q(v) + f(\pi(a),v)v\,;$

(iii) $\pi(a\zeta) = \overline{\pi(a)}q(\zeta) + \theta(a,\zeta) + f(\epsilon,\pi(a))\zeta\,;$

(iv) $\theta(av,u) = q(v)\overline{\theta(a,\overline{u})} + f(u,\overline{v})\overline{\theta(a,v)} + f(\theta(a,v),\overline{u})\overline{v} + \varphi(a,v)u\,;$

for all $a \in X$ *and all* $u,v \in V$.

Proof. By Lemma 2.7.134, this follows from the proof of [52, (26.30)]. □

Lemma 2.7.136. *For all* $v \in V$*, all* $w \in W$ *and all* $a \in X$*, we have that*

(i) $q(vw) = q(v)q(\epsilon w)\,;$

(ii) $q(\theta(a,v)) = q(v)q(\pi(a))\,.$

Proof. Since $\delta \cdot V \subseteq \operatorname{Rad}(H)$, it follows by substituting δ for w and w for z in $(\mathbf{Q}_{26})$ that $\delta \cdot \epsilon w \cdot v = \delta \cdot vw$, hence $[q(\epsilon w)] \cdot v = [q(vw)]$. By Lemma 2.7.51, it follows that $[q(v)q(\epsilon w)] = [q(vw)]$, which proves (i). Substituting $(a,0)$ for w in (i) now yields (ii). □

Lemma 2.7.137. *For all* $a \in X$*, we have that* $\varphi(a,\pi(a)) = 0$*.*

Proof. By Lemma 2.7.131, we may assume that $\operatorname{char}(K) = 2$. Since $f(\epsilon,\zeta) = 1 = q(\epsilon)$, we have that $q(\epsilon+\zeta) = q(\zeta)$. It then follows, by Lemma 2.7.124(iv) and Lemma 2.7.136(ii), that

$$\begin{aligned} q(\pi(a) + \theta(a,\zeta)) &= q(\theta(a,\epsilon+\zeta)) \\ &= q(\epsilon+\zeta)q(\pi(a)) \\ &= q(\zeta)q(\pi(a)) \\ &= q(\theta(a,\zeta))\,, \end{aligned}$$

and hence $q(\pi(a)) = f(\pi(a),\theta(a,\zeta))$. By Lemma 2.7.135(i), we now have that

$$\begin{aligned} \varphi(a,\pi(a)) &= f(\pi(a),\pi(a))f(\zeta,\pi(a)) + f(\theta(a,\zeta),\pi(a))f(\epsilon,\pi(a)) \\ &\qquad + f(\epsilon,\pi(a))q(\pi(a)) \\ &= 0 + q(\pi(a))f(\epsilon,\pi(a)) + f(\epsilon,\pi(a))q(\pi(a)) \\ &= 0\,, \end{aligned}$$

which is what we had to prove. □

Lemma 2.7.138. *For all $v \in V$ and all $w \in W^*$, we have that*

$$w \cdot q(\epsilon w)^{-1}\epsilon \cdot \overline{\epsilon w} \cdot vw = wv \ .$$

Proof. If we substitute $\lambda(w(-\epsilon))$ for w in ($\mathbf{Q}_{19}$), then we get that

$$\lambda(w(-\epsilon)) \cdot vw = wv \ .$$

If we set $v = \epsilon$ in this identity, then we get that $\lambda(w(-\epsilon)) \cdot \epsilon w = w$, hence $\lambda(w(-\epsilon)) = w \cdot (\epsilon w)^{-1}$, and therefore

$$w \cdot (\epsilon w)^{-1} \cdot vw = wv \ .$$

Note that it follows by substituting $[t]$ for z in ($\mathbf{Q}_{26}$), with $t \in K$, that $w \cdot t\epsilon \cdot v = w \cdot tv$. Since $(\epsilon w)^{-1} = q(\epsilon w)^{-1}\overline{\epsilon w}$, it follows from this identity that

$$w \cdot q(\epsilon w)^{-1}\epsilon \cdot \overline{\epsilon w} \cdot vw = wv \ ,$$

which is what we had to show. □

Lemma 2.7.139. *For all $a \in X$ and all $v \in V$, we have that*

(i) $q(\pi(a))av = a\overline{\pi(a)}\theta(a,v)$;

(ii) $a\pi(a)v = a\theta(a,v)$.

Proof. We may assume that $a \neq 0$. First, we substitute $(a,0)$ for w in Lemma 2.7.138, and we get that

$$(a,0) \cdot q(\epsilon(a,0))^{-1}\epsilon \cdot \overline{\epsilon(a,0)} \cdot v(a,0) = (a,0)v,$$

hence

$$(q(\pi(a))^{-1}a, 0) \cdot \overline{\pi(a)} \cdot \theta(a,v) = (a,0)v.$$

If we calculate the X-component of both sides, then we get that

$$q(\pi(a))^{-1}a\overline{\pi(a)}\theta(a,v) = av,$$

which shows (i).

On the other hand, if we substitute $(a,1)$ for w in Lemma 2.7.138, then we get that

$$(a,1) \cdot q(\epsilon(a,1))^{-1}\epsilon \cdot \overline{\epsilon(a,1)} \cdot v(a,1) = (a,1)v,$$

hence

$$(q(\pi(a)+\epsilon)^{-1}a, q(\pi(a)+\epsilon)^{-2}) \cdot (\overline{\pi(a)+\epsilon}) \cdot (\theta(a,v)+v) = (a,1)v.$$

Again, we calculate the X-component of both sides, and we get that

$$\big(q(\pi(a)) + f(\pi(a), \epsilon) + q(\epsilon)\big)^{-1} a(\overline{\pi(a)} + \epsilon)(\theta(a, v) + v) = av ,$$

from which it follows that

$$a\overline{\pi(a)}\theta(a, v) + a\theta(a, v) + a\overline{\pi(a)}v + av = q(\pi(a))av + f(\pi(a), \epsilon)av + av .$$

Since $f(\pi(a), \epsilon)av = a\pi(a)v + a\overline{\pi(a)}v$ by Lemma 2.7.128, it follows by (i) that

$$a\theta(a, v) = a\pi(a)v ,$$

which proves (ii). □

Lemma 2.7.140. *If $|K| > 2$, then*

$$c\theta(a, v) - c\pi(a)v = ah(a, c)v - ah(a, cv)$$

for all $a, c \in X$ and all $v \in V$.

Proof. See [52, (26.36)]. □

Lemma 2.7.141. *For all $a \in X$ and all $v \in V$, we have that*

$$\theta(a, \theta(a, v)) = \theta(a, v)f(\epsilon, \pi(a)) - q(\pi(a))v .$$

Proof. See [52, (26.33)]. □

We can rephrase this identity in terms of W in place of X, which results in a nice identity.

Lemma 2.7.142. *For all $v \in V$ and all $w \in W$, we have that*

$$vw(\boxminus w) = -q(\epsilon w)v .$$

Proof. Let $w = (a, t)$, with $a \in X$ and $t \in K$. Let $Q(a) := g(a, -a) = f(\pi(a), \epsilon)$, and observe that $2Q(a) = 0$; see Lemma 2.7.119 and Lemma 2.7.131. Note that, by Lemma 2.7.116, $\boxminus w = (-a, -t + Q(a))$. Hence, by Lemma 2.7.124 and Lemma 2.7.141,

$$\begin{aligned}
vw(\boxminus w) &= v(a, t)(-a, -t + Q(a)) \\
&= (\theta(a, v) + tv) \cdot (-a, -t + Q(a)) \\
&= \theta(-a, \theta(a, v) + tv) + (-t + Q(a))(\theta(a, v) + tv) \\
&= \theta(a, \theta(a, v)) + t\theta(a, v) - t\theta(a, v) + Q(a)\theta(a, v) - t^2 v + Q(a)tv \\
&= \theta(a, v)Q(a) - q(\pi(a))v + Q(a)\theta(a, v) - t^2 v + Q(a)tv \\
&= -\big(q(\pi(a)) + Q(a)t + t^2\big)v \\
&= -q\big(\pi(a) + t\epsilon\big)v \\
&= -q(\epsilon w)v ,
\end{aligned}$$

and we are done. □

Lemma 2.7.143. *For all $v \in V$ and all $w \in W$, we have that*

$$vww = f(\epsilon, \epsilon w)vw - q(\epsilon w)v \; .$$

Proof. By 2.2.13(i), we have that $w(-\epsilon) = F(\epsilon w, \epsilon) \boxminus w$, and hence, by $(\mathbf{Q}_6)$, $(\mathbf{Q}_{12})$ and Lemma 2.7.142,

$$\begin{aligned} vww &= vw \cdot w(-\epsilon) \\ &= vw \cdot (F(\epsilon w, \epsilon) \boxminus w) \\ &= vw \cdot [f(\epsilon w, \epsilon)] + vw(\boxminus w) \\ &= f(\epsilon w, \epsilon)vw - q(\epsilon w)v \; , \end{aligned}$$

which is what we had to show. □

Definition 2.7.144. For all $v \in V^*$ and all $w \in W^*$, we let $[v]_w := \langle v, vw \rangle$ be the subspace of V (over K) generated by v and vw. Note that $[v]_w$ is 2-dimensional if and only if $w \in W \setminus Y$.

Lemma 2.7.145. *For all $v \in V^*$ and all $w \in W \setminus Y$, we have $[v]_w \cdot w = [v]_w$, i.e. $[v]_w$ is a 2-dimensional subspace of V which is irreducible under the action of w.*

Proof. It follows from Lemma 2.7.143 that

$$\begin{aligned} [v]_w \cdot w &= \langle v, vw \rangle \cdot w = \langle vw, vww \rangle \\ &= \langle vw, f(\epsilon, \epsilon w)vw - q(\epsilon w)v \rangle = \langle vw, v \rangle \end{aligned}$$

since $q(\epsilon w) \neq 0$. □

Definition 2.7.146. Let $u, v \in V^*$ and $w \in W \setminus Y$. Then u and v are called *w-orthogonal* if and only if $f([u]_w, [v]_w) = 0$.

Remark 2.7.147. It is clear that the definition of $[v]_w$ and the notion of w-orthogonality are generalizations of the definition of $[v]_a$ and the notion of a-orthogonality as defined in [52]. See [52, (26.37) and (26.38)].

Theorem 2.7.148. *Let $a \in X^*$, and let $w := (a, 0) \in W^*$. Suppose that $f(\epsilon, \pi(a)) \neq 0$ if $\mathrm{char}(K) = 2$. Let T be the endomorphism of V given by $T(v) := vw$ for all $v \in V$. Then:*

(i) *The endomorphism T is a norm splitting map of the quadratic space (V, K, q);*

(ii) *The minimal polynomial of T is*

$$p(x) = x^2 + f(\epsilon, \pi(a))x + q(\pi(a)) .$$

Let E denote the splitting field of p over K, and let $\gamma \in E$ be a root of p. Then E/K is a separable quadratic extension and there is a scalar multiplication from $E \times V$ to V extending the scalar multiplication from $K \times V$ to V, such that $T(v) = \gamma v$ for all $v \in V$;

(iii) *Let S be a finite set of pairwise w-orthogonal elements of V^*. Then the elements of the set $S \cup Sw$ are linearly independent over K; if this set does not span V, then S can be extended to a larger set of non-zero pairwise w-orthogonal vectors ;*

(iv) *Let $\psi : E \longrightarrow [\epsilon]_w$ be given by*

$$\psi(r + t\gamma) := r\epsilon + t\pi(a)$$

for all $r, t \in K$. Then ψ is an isomorphism of vector spaces and X is a (right) vector space over E with scalar multiplication given by

$$bu := b\psi(u)$$

for all $b \in X$ and all $u \in E$. If σ denotes the non-trivial element in $\mathrm{Gal}(E/K)$, then $\psi(u^\sigma) = \overline{\psi(u)}$ for all $u \in E$. If N denotes the norm of the extension E/K, then $N(u) = q(\psi(u))$ for all $u \in E$.

Proof. See [52, (26.39)]. $\square$

Lemma 2.7.149. *Let $a \in X^*$ be arbitrary, and let $w := (a, 0) \in W^*$. Let $D := \langle \epsilon, \epsilon w, v, vw \rangle$ for some $v \in V \setminus \langle \epsilon, \epsilon w \rangle$. Then $\dim_K D = 4$, and we have that $aDD \subseteq aD$ (but not necessarily $bDD \subseteq bD$ for other elements $b \in X$).*

Proof. See [52, (26.41)]. $\square$

Theorem 2.7.150. *Let $\dim_K V = 4$. Then V can be made into a division ring such that X is a right vector space over V with scalar multiplication given by the map $(a, v) \mapsto av$ for all $a \in X$ and all $v \in V$.*

Proof. See [52, (26.42)]. $\square$

It will be convenient now to set $v^\sigma := \overline{v}$ for all $v \in V$.

Theorem 2.7.151. *Suppose that $\dim_K V \in \{2, 4\}$. Then there is a multiplication on V which gives V the structure of a division ring with the following properties:*

(i) $\langle\epsilon\rangle$ *is a subfield lying in the center of* V *and the map* $t \mapsto t\epsilon$ *is an isomorphism from* K *to* $\langle\epsilon\rangle$ *;*

(ii) σ *is an involution of* V *;*

(iii) X *is a right vector space over* V *with scalar multiplication given by the map* $(a, v) \mapsto av$ *;*

(iv) $q(v) = vv^\sigma = v^\sigma v \in \langle\epsilon\rangle$, *and* $f(u, v) = uv^\sigma + vu^\sigma = u^\sigma v + v^\sigma u \in \langle\epsilon\rangle$ *for all* $u, v \in V$ *;*

(v) h *is a skew-hermitian form on* X *with respect to* σ *;*

(vi) $(V, \langle\epsilon\rangle, \sigma)$ *is an involutory set ;*

(vii) $\theta(a, v) = \pi(a)v$ *for all* $a \in X$ *and all* $v \in V$.

Proof. See [52, (26.43)]. □

Theorem 2.7.152. *Suppose that* $\dim_K V \leq 4$. *Then* $\dim_K V \in \{2, 4\}$. *Let* V *be given the structure of a division ring as in Theorem 2.7.151. Then* (V, K, σ, X, π) *is an anisotropic pseudo-quadratic space. Moreover, we have that*

$$\pi(av) = v^\sigma \pi(a)v - \varphi(a, v)\epsilon$$

for all $a \in X$ *and all* $v \in V$.

Proof. See [52, (26.44)]. □

Theorem 2.7.153. *Suppose that* $\dim_K V \leq 4$. *Let* (V, K, σ, X, π) *be as in Theorem 2.7.152. Then* $(V, W, \tau_V, \tau_W, \epsilon, \delta) \cong \Omega_P(V, K, \sigma, X, \pi)$.

Proof. Let $(T, \boxplus)$ be the group defined in section 1.9.3 applied on the pseudo-quadratic space (V, K, σ, X, π). By the definition of the group T, we have that $\pi(a) - v \in \langle\epsilon\rangle$ for all $(a, v) \in T$. Let $\chi(a, v)$ be the unique element $t \in K$ such that $v - \pi(a) = t\epsilon$.

Let ϕ be the isomorphism from $[V]$ to V which maps $[v]$ to v for all $v \in V$, and let ψ be the isomorphism from $[T]$ to W which maps $[a, v]$ to $(a, \chi(a, v))$ for all $(a, v) \in T \subseteq X \times V$. Then $\phi([1]) = [1] = \epsilon$ and $\psi([0, 1]) = (0, \chi(0, 1)) = (0, 1) = \delta$ since $1\epsilon - \pi(0) = 1\epsilon$. (Remember that we have identified K with $\langle\epsilon\rangle \subseteq V$ by Theorem 2.7.151(i).)

Now, let $v \in V$ and $(a, x) \in T$ be arbitrary. By Lemma 2.7.139(ii) and Lemma 2.7.121(i), we have that

$$\begin{aligned}
a \cdot xv &= a\pi(a)v + axv - a\pi(a)v \\
&= a\theta(a, v) + a(x - \pi(a))v \\
&= a\theta(a, v) + a \cdot \chi(a, x)\epsilon \cdot v \\
&= a\theta(a, v) + \chi(a, x)av \\
&= a(\theta(a, v) + \chi(a, x)v) ,
\end{aligned}$$

and hence, by Theorem 2.7.151(iii), it follows that

$$xv = \theta(a, v) + \chi(a, x)v = v \cdot (a, \chi(a, x)) \ .$$

By Theorem 2.7.152 and Theorem 2.7.151(i and iv), we have that

$$\begin{aligned}
\chi(av, v^\sigma xv)\epsilon &= v^\sigma xv - \pi(av) \\
&= v^\sigma xv - v^\sigma \pi(a)v + \varphi(a, v)\epsilon \\
&= v^\sigma (x - \pi(a))v + \varphi(a, v)\epsilon \\
&= v^\sigma \cdot \chi(a, x)\epsilon \cdot v + \varphi(a, v)\epsilon \\
&= \big(\chi(a, x)q(v) + \varphi(a, v)\big)\epsilon \ ,
\end{aligned}$$

and hence

$$\chi(av, v^\sigma xv) = \chi(a, x)q(v) + \varphi(a, v) \ .$$

It follows that

$$\begin{aligned}
\phi([v][a, x]) &= \phi([xv]) = xv = v \cdot (a, \chi(a, x)) = \phi([v])\psi([a, x]) \ , \text{ and} \\
\psi([a, x][v]) &= \psi([av, v^\sigma xv]) = (av, \chi(av, v^\sigma xv)) \\
&= (av, \chi(a, x)q(v) + \varphi(a, v)) = (a, \chi(a, x)) \cdot v \\
&= \psi([a, x])\phi([v]) \ ,
\end{aligned}$$

for all $v \in V$ and all $(a, x) \in T$. Hence (ϕ, ψ) is an isomorphism from $\Omega_P(V, K, \sigma, X, \pi)$ to $(V, W, \tau_V, \tau_W, \epsilon, \delta)$. □

2.7.7 Quadrangular Systems of Type E_6, E_7 and E_8

In this section, we continue to assume that $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ is a wide quadrangular system which is the extension of a quadrangular system Λ of quadratic form type, such that $\mathrm{Rad}(F) = 0$. It only remains to consider the case where $\dim_K V > 4$.

Lemma 2.7.154. *If* $\mathrm{char}(K) = 2$*, then there exists an element* $\xi \in X^*$ *such that* $\pi(\xi) = \alpha\zeta$ *for some* $\alpha \in K^*$.

Proof. Suppose that $g(a, a) = 0$ for all $a \in X$. Since g is bilinear, it would follow that $g(a, b) = g(b, a)$ for all $a, b \in X$, and hence, by Lemma 2.7.116, that W is abelian. It would then follow by $(\mathbf{Q}_8)$ that $\mathrm{Im}(H) \subseteq \mathrm{Rad}(F)$. Since $\mathrm{Rad}(F) = 0$ and $H \not\equiv 0$, this is a contradiction.

Hence there exists an element $a \in X^*$ such that $g(a, a) \neq 0$. Let $w_1 := (a, 0) \in W^*$. By Lemma 2.7.119, it follows that $f(\epsilon, \epsilon w_1) = f(\epsilon, \pi(a)) = g(a, a) \neq 0$. Hence (see Definition 2.7.109)

$$S_1 \cap S_2 = \{v \in V \mid F(\epsilon, v) \neq 0\} \cap \{\epsilon w \mid w \in W\} \neq \emptyset \ ,$$

since $\epsilon w_1 \in S_1 \cap S_2$. By the definition of ζ, this implies that $\zeta = f(\epsilon, z)^{-1}z$ for some $z \in S_1 \cap S_2$. Let $z = \epsilon w_2$ for some $w_2 \in W^*$. Since $f(\epsilon w_2, \zeta) = f(z, \zeta) = f(z, f(\epsilon, z)^{-1}z) = 0$, w_2 is ζ-orthogonal, hence $w_2 = (\xi, 0)$ for some $\xi \in X^*$. We conclude that $\pi(\xi) = \epsilon(\xi, 0) = \epsilon w_2 = z = f(\epsilon, z)\zeta = \alpha\zeta$ for $\alpha = f(\epsilon, z) \in K^*$. □

Definition 2.7.155. If $\mathrm{char}(K) \neq 2$, let ξ be an arbitrary element of X^*. If $\mathrm{char}(K) = 2$, choose $\xi \in X^*$ as in Lemma 2.7.154.

By Lemma 2.7.154 and Theorem 2.7.148, the endomorphism T of V which maps v to $v(\xi, 0)$ is a norm splitting map of q.

We have come to a point which is completely similar to the beginning of Chapter 27 in [52], and the rest of the proof could literally be copied from that chapter.

Theorem 2.7.156. *The quadratic space (K, V_0, q) is of type E_6, E_7 or E_8.*

Proof. The proof is exactly as in [52, (27.17)], where we have to use Definition 2.7.117 and 2.7.118 and Lemmas 2.7.124, 2.7.127, 2.7.128, 2.7.129, 2.7.130, 2.7.131, 2.7.132, 2.7.135, 2.7.139, 2.7.140, 2.7.141 and 2.7.149. □

Theorem 2.7.157. $(V, W, \tau_V, \tau_W, \epsilon, \delta) \cong \Omega_E(K, V_0, q)$.

Proof. It follows from the proof of [52, (27.19)], using Definition 2.7.109 and 2.7.118 as well as Lemmas 2.7.124, 2.7.128, 2.7.129, 2.7.130, 2.7.131, 2.7.132, 2.7.133, 2.7.135 and 2.7.139, that the maps h, g, θ and φ are exactly as in section 2.6.5.

Let ϕ be the map from $[V_0]$ to V which maps $[v]$ to v for all $v \in V$, and let ψ be the map from $[S]$ to W which maps $[a, t]$ to (a, t) for all $(a, t) \in S$. Since we have seen in Definitions 2.7.117 and 2.7.118 that $(a, t) \cdot v = (av, tq(v) + \varphi(a, v))$ and $v \cdot (a, t) = \theta(a, v) + tv$, it is now obvious that (ϕ, ψ) is an isomorphism from $\Omega_E(K, V_0, q)$ to $(V, W, \tau_V, \tau_W, \epsilon, \delta)$. □

This completes the proof of Theorem 2.7.10, and thereby the proof of the classification of quadrangular systems.

2.8 Abelian Quadrangular Systems

In this last section of this chapter, we will describe the quadrangular systems $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ where W is abelian, and we will restate the axiom system for some specific cases.

A quadrangular system $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ will be called *abelian* if and only if W is abelian. One can check that Ω is abelian if and only if it is of quadratic form type, of involutory type, of indifferent type or of type F_4. (Note that, if Ω is of pseudo-quadratic form type with W abelian, then Ω is in fact reduced, and hence of one of these types.) In this case, we simply write $+$ and $-$ in place of $\boxplus$ and $\boxminus$, respectively, and we get the following description.

Consider an abelian group $(V, +)$ and an abelian group $(W, +)$. Suppose that there is a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W, both of which will be denoted by $\cdot$ or simply by juxtaposition, i.e. $\tau_V(v, w) = vw = v \cdot w$ and $\tau_W(w, v) = wv = w \cdot v$ for all $v \in V$ and all $w \in W$. Consider a map F from $V \times V$ to W and a map H from $W \times W$ to V which are additive in both variables. Suppose furthermore that there exists a fixed element $\epsilon \in V^*$ and a fixed element $\delta \in W^*$, and suppose that, for each $v \in V^*$, there exists an element $v^{-1} \in V^*$, and for each $w \in W^*$, there exists an element $w^{-1} \in W^*$, such that, for all $w, w_1, w_2 \in W$ and all $v, v_1, v_2 \in V$, the following axioms are satisfied.

(**A**$_1$) $w\epsilon = w$.

(**A**$_2$) $v\delta = v$.

(**A**$_3$) $(w_1 + w_2)v = w_1v + w_2v$.

(**A**$_4$) $(v_1 + v_2)w = v_1w + v_2w$.

(**A**$_5$) $v(-w) = -vw$.

(**A**$_6$) $w(-v) = wv$.

(**A**$_7$) $\mathrm{Im}(F) \subseteq \mathrm{Rad}(H)$.

(**A**$_8$) $\mathrm{Im}(H) \subseteq \mathrm{Rad}(F)$.

(**A**$_9$) $\delta \in \mathrm{Rad}(H)$.

(**A**$_{10}$) If $\mathrm{Rad}(F) \neq 0$, then $\epsilon \in \mathrm{Rad}(F)$.

(**A**$_{11}$) $w(v_1 + v_2) = wv_1 + wv_2 + F(v_1w, v_2)$.

(**A**$_{12}$) $v(w_1 + w_2) = vw_1 + vw_2 + H(w_1v, w_2)$.

(**A**$_{13}$) $(v^{-1})^{-1} = v$ (if $v \neq 0$).

(**A**$_{14}$) $(w^{-1})^{-1} = w$ (if $w \neq 0$).

(**A**$_{15}$) $wvv^{-1} = w$ (if $v \neq 0$).

(**A**$_{16}$) $vww^{-1} = v$ (if $w \neq 0$).

(**A**$_{17}$) $v^{-1}(wv) = \overline{vw}$ (if $v \neq 0$).

(**A**$_{18}$) $w^{-1}(vw) = wv$ (if $w \neq 0$).

(**A**$_{19}$) $F(v_1^{-1}, \overline{v_2})v_1 = F(v_1, v_2)$ (if $v_1 \neq 0$).

$(\mathbf{A}_{20})$ $H(w_1^{-1}, w_2)w_1 = H(w_1, w_2)$ (if $w_1 \neq 0$).

Then $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ is an abelian quadrangular system.

2.8.1 Reduced Quadrangular Systems

If $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ is *reduced* or *indifferent*, i.e. if $H \equiv 0$, then Ω is abelian, and we get the following description.

Consider an abelian group $(V, +)$ and an abelian group $(W, +)$. Suppose that there is a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W, both of which will be denoted by $\cdot$ or simply by juxtaposition, i.e. $\tau_V(v, w) = vw = v \cdot w$ and $\tau_W(w, v) = wv = w \cdot v$ for all $v \in V$ and all $w \in W$. Consider a map F from $V \times V$ to W which is additive in both variables. Suppose furthermore that there exists a fixed element $\epsilon \in V^*$ and a fixed element $\delta \in W^*$, and suppose that, for each $v \in V^*$, there exists an element $v^{-1} \in V^*$, and for each $w \in W^*$, there exists an element $w^{-1} \in W^*$, such that, for all $w, w_1, w_2 \in W$ and all $v, v_1, v_2 \in V$, the following axioms are satisfied.

$(\mathbf{R}_1)$ $w\epsilon = w$.

$(\mathbf{R}_2)$ $v\delta = v$.

$(\mathbf{R}_3)$ $(w_1 + w_2)v = w_1v + w_2v$.

$(\mathbf{R}_4)$ $(v_1 + v_2)w = v_1w + v_2w$.

$(\mathbf{R}_5)$ $v(-w) = -vw$.

$(\mathbf{R}_6)$ $w(-v) = wv$.

$(\mathbf{R}_7)$ If $\mathrm{Rad}(F) \neq 0$, then $\epsilon \in \mathrm{Rad}(F)$.

$(\mathbf{R}_8)$ $w(v_1 + v_2) = wv_1 + wv_2 + F(v_1w, v_2)$.

$(\mathbf{R}_9)$ $v(w_1 + w_2) = vw_1 + vw_2$.

$(\mathbf{R}_{10})$ $(v^{-1})^{-1} = v$ (if $v \neq 0$).

$(\mathbf{R}_{11})$ $(w^{-1})^{-1} = w$ (if $w \neq 0$).

$(\mathbf{R}_{12})$ $wvv^{-1} = w$ (if $v \neq 0$).

$(\mathbf{R}_{13})$ $vww^{-1} = v$ (if $w \neq 0$).

$(\mathbf{R}_{14})$ $v^{-1}(wv) = \overline{vw}$ (if $v \neq 0$).

$(\mathbf{R}_{15})$ $w^{-1}(vw) = wv$ (if $w \neq 0$).

$(\mathbf{R}_{16})$ $F(v_1^{-1}, \overline{v_2})v_1 = F(v_1, v_2)$ (if $v_1 \neq 0$).

Then $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ is a reduced or indifferent quadrangular system (and it is reduced if and only if $F \not\equiv 0$).

Remark 2.8.1. As we explained in Remark 2.5.4, axiom ($\mathbf{R}_7$) had only been introduced to simplify the classification result of the *wide* quadrangular systems. In particular, it is not needed for the reduced quadrangular systems, and it is actually often more convenient – and perfectly allowed – to leave it out.

2.8.2 Indifferent Quadrangular Systems

If $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ is *indifferent*, i.e. if $F \equiv 0$ and $H \equiv 0$, then Ω is abelian, and we get the following description.

Consider an abelian group $(V, +)$ and an abelian group $(W, +)$. Suppose that there is a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W, both of which will be denoted by $\cdot$ or simply by juxtaposition, i.e. $\tau_V(v, w) = vw = v \cdot w$ and $\tau_W(w, v) = wv = w \cdot v$ for all $v \in V$ and all $w \in W$. Suppose furthermore that there exists a fixed element $\epsilon \in V^*$ and a fixed element $\delta \in W^*$, and suppose that, for each $v \in V^*$, there exists an element $v^{-1} \in V^*$, and for each $w \in W^*$, there exists an element $w^{-1} \in W^*$, such that, for all $w, w_1, w_2 \in W$ and all $v, v_1, v_2 \in V$, the following axioms are satisfied.

($\mathbf{D}_1$) $w\epsilon = w$.

($\mathbf{D}_2$) $v\delta = v$.

($\mathbf{D}_3$) $(w_1 + w_2)v = w_1v + w_2v$.

($\mathbf{D}_4$) $(v_1 + v_2)w = v_1w + v_2w$.

($\mathbf{D}_5$) $w(v_1 + v_2) = wv_1 + wv_2$.

($\mathbf{D}_6$) $v(w_1 + w_2) = vw_1 + vw_2$.

($\mathbf{D}_7$) $(v^{-1})^{-1} = v$ (if $v \neq 0$).

($\mathbf{D}_8$) $(w^{-1})^{-1} = w$ (if $w \neq 0$).

($\mathbf{D}_9$) $wvv^{-1} = w$ (if $v \neq 0$).

($\mathbf{D}_{10}$) $vww^{-1} = v$ (if $w \neq 0$).

($\mathbf{D}_{11}$) $v^{-1}(wv) = vw$ (if $v \neq 0$).

($\mathbf{D}_{12}$) $w^{-1}(vw) = wv$ (if $w \neq 0$).

Then $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ is an indifferent quadrangular system. (Note that we do not have to assume *a priori* that all elements of V and W have order at most 2, but that this follows from these axioms.)

2.8.3 Radical Quadrangular Systems

An abelian quadrangular system $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ will be called *radical* if and only if $\mathrm{Rad}(F) \neq 0$. One can check that Ω is radical if and only if it is of quadratic form type with $\epsilon \in \mathrm{Rad}(f)$ (and hence $\mathrm{char}(K) = 2$), of indifferent type or of type F_4. We will give two different (but equivalent) descriptions. The first one is useful to check whether a certain system is a radical quadrangular system; the second one is more convenient to work with. Note that each of these descriptions is completely symmetrical.

First Description

Consider an abelian group $(V, +)$ and an abelian group $(W, +)$. Suppose that there is a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W, both of which will be denoted by $\cdot$ or simply by juxtaposition, i.e. $\tau_V(v, w) = vw = v \cdot w$ and $\tau_W(w, v) = wv = w \cdot v$ for all $v \in V$ and all $w \in W$. Consider a map F from $V \times V$ to W and a map H from $W \times W$ to V which are additive in both variables. Suppose furthermore that there exists a fixed element $\epsilon \in V^*$ and a fixed element $\delta \in W^*$, and suppose that, for each $v \in V^*$, there exists an element $v^{-1} \in V^*$, and for each $w \in W^*$, there exists an element $w^{-1} \in W^*$, such that, for all $w, w_1, w_2 \in W$ and all $v, v_1, v_2 \in V$, the following axioms are satisfied.

($\mathbf{F}_1$) $w\epsilon = w$.

($\mathbf{F}_2$) $v\delta = v$.

($\mathbf{F}_3$) $(w_1 + w_2)v = w_1 v + w_2 v$.

($\mathbf{F}_4$) $(v_1 + v_2)w = v_1 w + v_2 w$.

($\mathbf{F}_5$) $\mathrm{Im}(F) \subseteq \mathrm{Rad}(H)$.

($\mathbf{F}_6$) $\mathrm{Im}(H) \subseteq \mathrm{Rad}(F)$.

($\mathbf{F}_7$) $\delta \in \mathrm{Rad}(H)$.

($\mathbf{F}_8$) $\epsilon \in \mathrm{Rad}(F)$.

($\mathbf{F}_9$) $w(v_1 + v_2) = wv_1 + wv_2 + F(v_1 w, v_2)$.

($\mathbf{F}_{10}$) $v(w_1 + w_2) = vw_1 + vw_2 + H(w_1 v, w_2)$.

($\mathbf{F}_{11}$) $(v^{-1})^{-1} = v$ (if $v \neq 0$).

($\mathbf{F}_{12}$) $(w^{-1})^{-1} = w$ (if $w \neq 0$).

($\mathbf{F}_{13}$) $wvv^{-1} = w$ (if $v \neq 0$).

($\mathbf{F}_{14}$) $vww^{-1} = v$ (if $w \neq 0$).

($\mathbf{F}_{15}$) $v^{-1}(wv) = vw$ (if $v \neq 0$).

($\mathbf{F}_{16}$) $w^{-1}(vw) = wv$ (if $w \neq 0$).

($\mathbf{F}_{17}$) $F(v_1^{-1}, v_2)v_1 = F(v_1, v_2)$ (if $v_1 \neq 0$).

($\mathbf{F}_{18}$) $H(w_1^{-1}, w_2)w_1 = H(w_1, w_2)$ (if $w_1 \neq 0$).

Then $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ is a radical quadrangular system. It is of type F_4 if and only if $F \not\equiv 0$ and $H \not\equiv 0$.

Second Description

Let K and L be two commutative fields with $\mathrm{char}(K) = \mathrm{char}(L) = 2$, such that K is a vector space over L and that L is a vector space over K. If t is an element of the field K, then we will denote the corresponding element of the vector space K by $[t]$; if s is an element of the field L, then we will denote the corresponding element of the vector space L by $[s]$. Let V be a vector space over K containing $[L]$ as a subspace, and let W be a vector space over L containing $[K]$ as a subspace.

Suppose that q is an anisotropic quadratic form from V to K, with corresponding bilinear form f, and that $\hat{q}$ is an anisotropic quadratic form from W to L, with corresponding bilinear form $\hat{f}$, such that $[L] \subseteq \mathrm{Rad}(f)$ and $[K] \subseteq \mathrm{Rad}(\hat{f})$. Let $\epsilon := [1] \in [L] \subseteq V$ and $\delta := [1] \in [K] \subseteq W$. Finally, suppose that there is a map τ_V from $V \times W$ to V which is K-linear on V, and a map τ_W from $W \times V$ to W which is L-linear on W, both of which will be denoted by $\cdot$ or simply by juxtaposition, i.e. $\tau_V(v, w) = vw = v \cdot w$ and $\tau_W(w, v) = wv = w \cdot v$ for all $v \in V$ and all $w \in W$. Moreover, suppose that the following axioms hold, for all $v \in V$, $w \in W$, $t \in K$ and $s \in L$.

($\mathbf{C}_1$) $v[t] = tv$.

($\mathbf{C}_2$) $w[s] = sw$.

($\mathbf{C}_3$) $v \cdot sw = vw \cdot s\delta$.

($\mathbf{C}_4$) $w \cdot tv = wv \cdot t\epsilon$.

($\mathbf{C}_5$) $[t]v = [tq(v)]$.

($\mathbf{C}_6$) $[s]w = [s\hat{q}(w)]$.

($\mathbf{C}_7$) $vww = v \cdot \hat{q}(w)\delta$.

($\mathbf{C}_8$) $wvv = w \cdot q(v)\epsilon$.

($\mathbf{C}_9$) $v \cdot wv = q(v)vw$.

($\mathbf{C}_{10}$) $w \cdot vw = \hat{q}(w)wv$.

($\mathbf{C}_{11}$) $v(w_1 + w_2) = vw_1 + vw_2 + [\hat{f}(w_1 v, w_2)]$.

($\mathbf{C}_{12}$) $w(v_1 + v_2) = wv_1 + wv_2 + [f(v_1 w, v_2)]$.

Then $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ is a radical quadrangular system. It is of type F_4 if and only if $f \not\equiv 0$ and $\hat{f} \not\equiv 0$.

3 Automorphisms of F_4 Quadrangles

An important problem in the study of Moufang polygons is to determine the structure of the automorphism group G modulo the subgroup $G^\dagger$ generated by all the root groups. In [52], this has been done for four of the six different families of Moufang quadrangles. The two classes which have been left open, are the cases of the exceptional quadrangles, that is, those of type E_6, E_7 and E_8, and those of type F_4.

The goal of this chapter is to determine the quotient $G/G^\dagger$ for the latter case. More precisely, we will show that the automorphism group is, up to field automorphisms, generated by the root groups. In order to obtain this result, we will use the quadrangular systems which we have introduced in Chapter 2. In particular, we will make use of the second description of a quadrangular system of type F_4 as given in section 2.8.3.

3.1 Main Theorem

Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be an arbitrary quadrangular system of type F_4, and let $\Gamma := \mathcal{Q}(\Omega)$ be the corresponding Moufang quadrangle. Let $\mathrm{Cor}(\Gamma)$ be the full correlation group of Γ, and let $G := \mathrm{Aut}(\Gamma)$ be its subgroup of type-preserving automorphisms. Then G is of index at most 2 in $\mathrm{Cor}(\Gamma)$. Let $G^\dagger$ be the subgroup of G generated by all the root groups of Γ.

Let K and L be as in section 2.6.6. In particular, L is a subfield of K. Let $\mathrm{Aut}(K, L)$ denote the group of field automorphisms of K which map L to itself. Note that it follows from the fact that $\mathrm{char}(K) = \mathrm{char}(L) = 2$ that $\mathrm{Aut}(K, L) \cong \mathrm{Aut}(L, K^2)$.

We can now state the Main Theorem of this chapter.

Main Theorem 3.1.1. *$G/G^\dagger$ is isomorphic to a subgroup of* $\mathrm{Aut}(K, L)$.

Remark 3.1.2. $[\mathrm{Cor}(\Gamma) : G]$ is 1 or 2, and both cases can actually occur. By [52, (35.12)], Γ has a correlation if and only if the quadratic space (K, V, q) is similar to $(L, W, \hat{q})$ as defined on page 12. In [52, (14.25) and (14.26)], examples are given of quadratic spaces of type F_4 which do have this property and others which do not.

3.2 Similarities

We will first translate the geometric problem of isomorphic Moufang quadrangles of type F_4 to the algebraic problem of similar quadrangular systems of type F_4.

Definition 3.2.1. Let Ω and Ω' be two quadrangular systems of type F_4, and write $\Omega := (V, W, \tau_V, \tau_W, \epsilon, \delta)$ and $\Omega' := (V', W', \tau_{V'}, \tau_{W'}, \epsilon', \delta')$. We will say that $(\varphi, \hat{\varphi})$ is a *similarity* from Ω to Ω' if and only if φ and $\hat{\varphi}$ are group isomorphisms from $(V, +)$ to $(V', +)$ and from $(W, +)$ to $(W', +)$, respectively, for which there exist constants $g \in K'^*$ and $\hat{g} \in L'^*$ (called the *parameters* of the similarity) such that

$$\varphi(vw) = g\varphi(v)\hat{\varphi}(w) \tag{3.1}$$

$$\hat{\varphi}(wv) = \hat{g}\hat{\varphi}(w)\varphi(v) \tag{3.2}$$

for all $v \in V$ and all $w \in W$. A similarity from Ω to itself will be called a *self-similarity*. Moreover, if both φ and $\hat{\varphi}$ are vector space isomorphisms, then the self-similarity $(\varphi, \hat{\varphi})$ will be called *linear*.

Remark 3.2.2. If $(\varphi_1, \hat{\varphi}_1)$ and $(\varphi_2, \hat{\varphi}_2)$ are two self-similarities of Ω with parameters $(g_1, \hat{g}_1)$ and $(g_2, \hat{g}_2)$, respectively, then their product $(\varphi_1\varphi_2, \hat{\varphi}_1\hat{\varphi}_2)$ is again a self-similarity. If both $(\varphi_1, \hat{\varphi}_1)$ and $(\varphi_2, \hat{\varphi}_2)$ are linear, then their product is also linear, and has parameters $(g_1g_2, \hat{g}_1\hat{g}_2)$.

Theorem 3.2.3. *Let Ω and Ω' be two quadrangular systems of type F_4. Let $\mathcal{Q}(\Omega)$ and $\mathcal{Q}(\Omega')$ be the corresponding Moufang quadrangles with labeled base apartments $\Sigma = \{0, \dots, 7\}$ and $\Sigma' = \{0', \dots, 7'\}$, respectively. Let $H_{\Sigma,\Sigma'}$ denote the set of isomorphisms from $\mathcal{Q}(\Omega)$ to $\mathcal{Q}(\Omega')$ mapping i to i' for all $i \in \Sigma$, and let $X_{\Omega,\Omega'}$ denote the set of similarities from Ω to Ω'. Then there is a natural one-to-one correspondence between $H_{\Sigma,\Sigma'}$ and $X_{\Omega,\Omega'}$ which is a group isomorphism if $\Omega = \Omega'$ and $\Sigma = \Sigma'$. In particular, $\mathcal{Q}(\Omega)$ and $\mathcal{Q}(\Omega')$ are isomorphic (in the type-preserving sense) if and only if Ω and Ω' are similar.*

Proof. For every object $\mathfrak{o}$ which we have defined for Ω, we will denote the corresponding object in Ω' by $\mathfrak{o}'$. For example, we will use the notations U_i', $\hat{f}'$, K', and so on.

Let $U_+ := \langle U_1, \ldots, U_4 \rangle$ and $U'_+ := \langle U'_1, \ldots, U'_4 \rangle$, and let Y denote the set of isomorphisms from U_+ to U'_+ mapping U_i to U'_i for all $i \in \{1, \ldots, 4\}$. By Theorem 1.4.8, there is a natural one-to-one correspondence between Y and $H_{\Sigma,\Sigma'}$, which is a group isomorphism if $\Omega = \Omega'$ and $\Sigma = \Sigma'$.

A collection of group isomorphisms $\phi_i : U_i \to U'_i$ with $i \in \{1, \ldots, 4\}$ induces an element of Y (which we will then denote by $y(\phi_1, \ldots, \phi_4)$) if and only if it preserves the commutator relations. By the relations (2.1) on page 37, this amounts to the conditions

$$\phi_2([\hat{f}(w_1, w_2)]) = [\hat{f}'(\phi_1(w_1), \phi_3(w_2))] \;, \tag{3.3}$$
$$\phi_3([f(v_1, v_2)]) = [f'(\phi_2(v_1), \phi_4(v_2))] \;, \tag{3.4}$$
$$\phi_2(vw) = \phi_4(v)\phi_1(w) \;, \tag{3.5}$$
$$\phi_3(wv) = \phi_1(w)\phi_4(v) \;, \tag{3.6}$$

for all $v, v_1, v_2 \in V$ and all $w, w_1, w_2 \in W$. We will first show that (3.3) and (3.4) follow from (3.5) and (3.6). So assume that (3.5) and (3.6) hold, for all $v \in V$ and all $w \in W$. Then, by $(\mathbf{C}_{11})$,

$$\begin{aligned}
\phi_2([\hat{f}(w_1, w_2)]) &= \phi_2\big(\epsilon(w_1 + w_2) + \epsilon w_1 + \epsilon w_2\big) \\
&= \phi_4(\epsilon)\phi_1(w_1 + w_2) + \phi_4(\epsilon)\phi_1(w_1) + \phi_4(\epsilon)\phi_1(w_2) \\
&= [\hat{f}'(\phi_1(w_1), \phi_1(w_2)\phi_4(\epsilon))] \\
&= [\hat{f}'(\phi_1(w_1), \phi_3(w_2))]
\end{aligned}$$

for all $w_1, w_2 \in W$, which shows (3.3). The proof of (3.4) is similar.

Now, assume first that $(\varphi, \hat{\varphi}) \in X_{\Omega,\Omega'}$ is a similarity from Ω to Ω', with parameters $g \in K^*$ and $\hat{g} \in L^*$. Let $\phi_1(w) := \hat{\varphi}(w)$, $\phi_2(v) := g^{-1}\varphi(v)$, $\phi_3(w) := \hat{g}^{-1}\hat{\varphi}(w)$ and $\phi_4(v) := \varphi(v)$, for all $v \in V$ and all $w \in W$. Then it follows immediately from (3.1) and (3.2) that (3.5) and (3.6) hold; hence $(\phi_1, \ldots, \phi_4)$ induce an element

$$y(\varphi, \hat{\varphi}) := y(\hat{\varphi}, g^{-1}\varphi, \hat{g}^{-1}\hat{\varphi}, \varphi) \in Y \;. \tag{3.7}$$

We have thus defined an injective map y from $X_{\Omega,\Omega'}$ to Y. Observe that it follows from (3.7) that y is a group homomorphism if $\Omega = \Omega'$.

It remains to show that the map y is onto. So let $z \in Y$ be arbitrary, and let $\phi_1, \ldots, \phi_4$ denote the restriction of z to the groups $U_1, \ldots, U_4$, respectively. Then (3.3) – (3.6) hold. It follows from (3.3) that $\phi_1(\mathrm{Rad}(\hat{f})) = \mathrm{Rad}(\hat{f}') = [K']$; hence $\phi_1(\delta) = [g]$ for some $g \in K'^*$. Similarly, it follows from (3.4) that $\phi_4(\epsilon) = [\hat{g}]$ for some $\hat{g} \in L'^*$. If we substitute δ for w in (3.5), then we get that $\phi_2(v) = g\phi_4(v)$ for all $v \in V$, and hence $\phi_4(vw) =$

$g^{-1}\phi_4(v)\phi_1(w)$. Similarly, we have that $\phi_1(wv) = \hat{g}^{-1}\phi_1(w)\phi_4(v)$. Therefore (ϕ_4, ϕ_1) is a similarity from Ω to Ω' with parameters $(g^{-1}, \hat{g}^{-1})$, and $z = y(\phi_4, \phi_1)$.

We conclude that there is a natural one-to-one correspondence between Y and $X_{\Omega,\Omega'}$, which is a group isomorphism if $\Omega = \Omega'$ and $\Sigma = \Sigma'$.

Finally, it follows from Theorem 1.4.10 that there exists a type-preserving isomorphism from $\mathcal{Q}(\Omega)$ to $\mathcal{Q}(\Omega')$ if and only if there exists an isomorphism in $H_{\Sigma,\Sigma'}$ from $\mathcal{Q}(\Omega)$ to $\mathcal{Q}(\Omega')$. The last statement now follows from the fact that $H_{\Sigma,\Sigma'} \neq \emptyset$ if and only if $X_{\Omega,\Omega'} \neq \emptyset$. □

As before, let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system of type F_4, and let H be the pointwise stabilizer of the labeled base apartment $\Sigma = \{0, \dots, 7\}$ of $\mathcal{Q}(\Omega)$. Let X denote the group of self-similarities of Ω, and let X_ℓ denote its subgroup of linear self-similarities.

Lemma 3.2.4. *$H \cong X$.*

Proof. This follows from Theorem 3.2.3 with $\Omega = \Omega'$ and $\Sigma = \Sigma'$. □

3.3 Multipliers of Similitudes of q

From now on, we will always assume that $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ is a quadrangular system of type F_4. The first step towards the examination of the automorphisms is the determination of the multipliers of the similitudes of q.

The following lemma is crucial. We use a technique similar to the one in the proof of [52, (14.17)].

Lemma 3.3.1. *Let $a, d \in V \setminus [L]$ be arbitrary elements such that $f(a, d) = 0$. Let $\gamma := q(a)/q(d)$. If $\gamma \in L$, then $\gamma \in K^2 \cdot \hat{q}(W)$.*

Proof. Let $\xi \in W \setminus [K]$ be arbitrary, and let $e \in V$ be as in Theorem 2.7.98. As in section 2.7.5, we define $\alpha := \hat{q}(\xi) \in L \setminus K^2$ and $\beta := q(d)^{-1} \in K \setminus L$. By Theorems 2.7.99 and 2.7.100, there exist elements $t_1, t_2, t_3, t_4 \in K$ and $s \in L$ such that $a = t_1 d + t_2 e + t_3 d\xi + t_4 e\xi + [s]$. Since $f(d, d) = f(d, d\xi) = f(d, e\xi) = 0$ and $f(d, e) = 1$, it follows from $f(a, d) = 0$ that $t_2 = 0$. Hence, by Theorem 2.7.103(i),

$$\begin{aligned}
\gamma &= q(a)/q(d) \\
&= \beta q(t_1 d + t_3 d\xi + t_4 e\xi + [s]) \\
&= \big(t_1^2 + \alpha N(t_3 + t_4\omega)\big) + \beta s \,.
\end{aligned}$$

Since $\gamma \in L$, it follows that $N(t_3 + t_4\omega) \in L(\beta)$. By [52, (14.9)], we know that $L(\beta) \cap N(E) = K^2 \cdot N(D(\beta))$. In particular,

$$N(t_3 + t_4\omega) = \lambda^2 N(x + \beta y) = \lambda^2\big(N(x) + \beta^2 N(y) + \beta(x\bar{y} + \bar{x}y)\big)$$

for some $\lambda \in K$ and some $x, y \in D$. Hence

$$\gamma = t_1^2 + \alpha\lambda^2\big(N(x) + \beta^2 N(y)\big) + \beta\big(s + \lambda^2(x\bar{y} + \bar{x}y)\big) \ .$$

Since $\gamma \in L$, $t_1^2 + \alpha\lambda^2\big(N(x) + \beta^2 N(y)\big) \in L$ and $\big(s + \lambda^2(x\bar{y} + \bar{x}y)\big) \in L$, but $\beta \notin L$, it follows that $\big(s + \lambda^2(x\bar{y} + \bar{x}y)\big) = 0$, hence

$$\gamma = t_1^2 + \alpha\lambda^2\big(N(x) + \beta^2 N(y)\big) \ .$$

If $\lambda = 0$, this implies that $\gamma = t_1^2 = t_1^2\hat{q}(\delta) \in K^2 \cdot \hat{q}(W)$; if $\lambda \neq 0$, it follows that

$$\gamma = \lambda^2 \cdot \Big(\alpha\big(N(x) + \beta^2 N(y)\big) + (\lambda^{-1}t_1)^2\Big) \in K^2 \cdot \hat{q}(W) \ .$$

□

Lemma 3.3.2. *Let $a, d \in V$ be arbitrary elements such that $f(a, d) \neq 0$. Then there exists an element $w \in W$ such that $f(aw, d) = 0$.*

Proof. As in the previous lemma, let $\xi \in W \setminus [K]$ be arbitrary, and let $e \in V$ be as in Theorem 2.7.98; then there exist elements $t_1, t_2, t_3, t_4 \in K$ and $s \in L$ such that $a = t_1 d + t_2 e + t_3 d\xi + t_4 e\xi + [s]$. Since $f(a, d) = t_2$, it follows that $t_2 \neq 0$. Let $w := \xi + [\alpha t_4 t_2^{-1}] \in W$. Then, by $(\mathbf{C}_{11})$, $dw = d\xi + \alpha t_4 t_2^{-1} d$, since $[\alpha t_4 t_2^{-1}] \in \mathrm{Rad}(\hat{f})$. Hence

$$\begin{aligned} f(aw, d) &= f(dw, a) = f(d\xi, a) + \alpha t_4 t_2^{-1} f(d, a) \\ &= t_4 f(d\xi, e\xi) + \alpha t_4 t_2^{-1} t_2 = t_4\alpha + \alpha t_4 = 0 \ , \end{aligned}$$

which is what we had to show. □

Theorem 3.3.3. $G(q) = K^2 \cdot \hat{q}(W) \cdot \hat{q}(W) \setminus \{0\}$.

Proof. Let T be an arbitrary similitude of q, with multiplier $\lambda \in K^*$. Then $q(Tv) = \lambda q(v)$ for all $v \in V$. In particular, if $v \in \mathrm{Rad}(f)$, then we have $Tv \in \mathrm{Rad}(f)$ as well; hence $T\epsilon \in \mathrm{Rad}(f) = [L]$. Since $q(\epsilon) = 1$, it follows that $\lambda = q(T\epsilon) \in q([L]) = L$.

Again, let $d \in V \setminus [L]$ and $\xi \in W \setminus [K]$ be arbitrary. If $f(d, Td) = 0$, then it follows from Lemma 3.3.1 that $\lambda = q(Td)/q(d) \in K^2 \cdot \hat{q}(W)$, and we are done.

So we can assume that $f(d, Td) \neq 0$. Then it follows from Lemma 3.3.2 that there exists an element $w \in W$ such that $f(dw, Td) = 0$. Now consider the maps $T_1 : V \to V : v \mapsto vw$ and $T_2 := T \circ T_1^{-1}$. Then T_1 is a similitude with multiplier $\hat{q}(w)$, since $q(vw) = \hat{q}(w)q(v)$ for all $v \in V$ by Lemmas 2.7.95(ii) and 2.7.97(i). It follows that T_2 is a similitude as well, with multiplier $\lambda_2 := \lambda\hat{q}(w)^{-1}$. Then $T = T_2 \circ T_1$, and we have that $f(dw, T_2(dw)) = f(dw, Td) = 0$. By the previous paragraph with T_2 in place of T and dw in place of d, we get that $\lambda_2 \in K^2 \cdot \hat{q}(W)$, and it finally follows that $\lambda = \lambda_2\hat{q}(w) \in K^2 \cdot \hat{q}(W) \cdot \hat{q}(W)$.

On the other hand, for every $t \in K^*$ and every $w_1, w_2 \in W^*$, we can find a similitude with multiplier $t^2\hat{q}(w_1)\hat{q}(w_2)$, namely

$$T_{t,w_1,w_2} : V \to V : v \mapsto tv \cdot w_1 \cdot w_2 \ ,$$

and we are done. □

3.4 Action on the Root Groups

We continue to assume that Ω is a quadrangular system of type F_4. Observe that the reflections π_c and Π_z as defined on page 26 are now just ordinary reflections with respect to the quadratic forms q and $\hat{q}$ as defined on page 11. In particular, we will write $\hat{\pi}_z$ in place of Π_z for all $z \in W^*$. The identities $(\mathbf{Q}_{23})$ – $(\mathbf{Q}_{26})$ translate to the following properties.

Lemma 3.4.1. *For all $v \in V$, $c \in V^*$, $w \in W$ and $z \in W^*$, we have that*

(i) $w\pi_c(v) = wc^{-1}vc\,;$

(ii) $v\hat{\pi}_z(w) = vz^{-1}wz\,;$

(iii) $\pi_c(vw) = q(c)^{-1}\pi_c(v) \cdot wc\,;$

(iv) $\hat{\pi}_z(wv) = \hat{q}(z)^{-1}\hat{\pi}_z(w) \cdot vz\,.$

By (1.1) on page 10, the problem of determining the quotient $G/G^\dagger$ is reduced to the determination of $H/H^\dagger$, where H is the pointwise stabilizer of the labeled base apartment $\Sigma = \{0, \dots, 7\}$ of $\mathcal{Q}(\Omega)$ and $H^\dagger = H \cap G^\dagger$. By Theorem 1.4.11(i), H acts faithfully on $U_1 \times U_4$, so the whole problem is reduced to the examination of the action of H on the root groups U_1 and U_4.

Using Theorem 1.4.11(ii), we can now proceed by explicitly calculating the action of $H^\dagger$ on $U_1 \times U_4$. Very similarly as in [52, (33.11)], we can compute the action of the elements $\mu_1^{(z)} := \mu(x_1(\delta))\mu(x_1(z))$ and

$\mu_4^{(c)} := \mu(x_4(\epsilon))\mu(x_4(c))$ on $U_1 \times U_4$. Note that it follows from 1.4.11(ii) that

$$H^\dagger = \left\langle \mu_1^{(z)}, \mu_4^{(c)} \mid z \in W, c \in V \right\rangle . \tag{3.8}$$

We will only calculate the action of $\mu_4^{(c)}$, the action of $\mu_1^{(z)}$ being completely similar by the symmetric structure of the quadrangular systems of type F_4. So let c be a fixed arbitrary element of V^*. By Theorem 2.3.2, we have that

$$\begin{aligned}
x_1(w)^{\mu_4^{(c)}} &= x_1(w)^{\mu(x_4(\epsilon))\mu(x_4(c))} \\
&= x_3(w)^{\mu(x_4(c))} = x_1(wc^{-1}) ; \\
x_2(v)^{\mu_4^{(c)}} &= x_2(v)^{\mu(x_4(\epsilon))\mu(x_4(c))} \\
&= x_2(v)^{\mu(x_4(c))} = x_2(\pi_c(v)) ; \\
x_3(w)^{\mu_4^{(c)}} &= x_3(w)^{\mu(x_4(\epsilon))\mu(x_4(c))} \\
&= x_1(w)^{\mu(x_4(c))} = x_3(wc) ;
\end{aligned}$$

for all $v \in V$ and all $w \in W$. Now let $v \in V$ be arbitrary; by condition $(\mathcal{M}_3)$ of Theorem 1.4.4, $U_4^{\mu_4^{(c)}} = U_4$, hence there exists an element $u \in V$ such that $x_4(v)^{\mu_4^{(c)}} = x_4(u)$. We now let $\mu_4^{(c)}$ act on both sides of the commutator relation

$$[x_1(\delta), x_4(v)] = x_2(v)x_3(\delta v)$$

to get that

$$[x_1(\delta c^{-1}), x_4(u)] = x_2(\pi_c(v))x_3(\delta v \cdot c) .$$

On the other hand,

$$[x_1(\delta c^{-1}), x_4(u)] = x_2(u \cdot \delta c^{-1})x_3(\delta c^{-1} \cdot u) ,$$

and comparing the U_2-components of both expressions yields

$$u \cdot \delta c^{-1} = \pi_c(v) .$$

Since $u \cdot \delta c^{-1} = q(c)^{-1}u$ by $(\mathbf{C}_5)$ and $(\mathbf{C}_1)$, it follows that $u = q(c)\pi_c(v)$. We conclude that the action of $\mu_4^{(c)}$ on $U_1 \times U_4$ corresponds to the map

$$\theta_c : (w, v) \mapsto (wc^{-1}, q(c)\pi_c(v))$$

from $W \times V$ to itself. Completely similarly, we obtain that the action of $\mu_1^{(z)}$ on $U_1 \times U_4$ corresponds to the map

$$\hat{\theta}_z : (w, v) \mapsto (\hat{q}(z)\hat{\pi}_z(w), vz^{-1})$$

from $W \times V$ to itself. The maps $\hat{\theta}_z$ and θ_c can also be interpreted as self-similarities of Ω. By (3.8) and the definition of the isomorphism y in (3.7), we thus get that

$$H^\dagger \cong X^\dagger := \left\langle \hat{\theta}_z, \theta_c \mid z \in W^*, c \in V^* \right\rangle \ .$$

Note that it now follows from Lemma 3.2.4 that

$$H/H^\dagger \cong X/X^\dagger \ . \tag{3.9}$$

We will now define some useful elements in $X^\dagger$.

Definition 3.4.2. For all $z \in W^*$ and all $c \in V^*$, we define the self-similarities

$$\hat{\chi}_z := \theta_{[\hat{q}(z)]}\hat{\theta}_z \qquad \text{and} \qquad \chi_c := \hat{\theta}_{[q(c)]}\theta_c \ .$$

It follows from the fact that $\pi_{[s]} = 1$ and $\hat{\pi}_{[t]} = 1$ for all $s \in L^*$ and all $t \in K^*$ that

$$\begin{aligned} \hat{\chi}_z(v, w) &= (vz, \hat{\pi}_z(w)) \ , \\ \chi_c(v, w) &= (\pi_c(v), wc) \ , \end{aligned}$$

for all $c \in V^*, z \in W^*$ and all $v \in V, w \in W$.

Lemma 3.4.3. *For all $z \in W^*$, $\hat{\chi}_z$ is a self-similarity of Ω with parameters $(1, \hat{q}(z)^{-1})$; for all $c \in V^*$, χ_c is a self-similarity of Ω with parameters $(q(c)^{-1}, 1)$.*

Proof. We will only show the second statement, the first one being completely similar. So let $c \in V^*$ be arbitrary, and let $\varphi : V \to V : v \mapsto \pi_c(v)$ and $\hat{\varphi} : W \to W : w \mapsto wc$. Then, by Lemma 3.4.1(i and iii),

$$\begin{aligned} \varphi(vw) &= \pi_c(vw) = q(c)^{-1}\pi_c(v) \cdot wc = q(c)^{-1}\varphi(v)\hat{\varphi}(w) \ , \\ \hat{\varphi}(wv) &= wvc = (wc)c^{-1}vc = wc \cdot \pi_c(v) = \hat{\varphi}(w)\varphi(v) \ , \end{aligned}$$

for all $v \in V$ and all $w \in W$, and we are done. $\square$

3.4.1 Field Automorphisms

Recall that X_ℓ is the subgroup of X consisting of the linear self-similarities, and that $\mathrm{Aut}(K, L)$ is the group of field automorphisms of K which map L to itself. We will first show that X/X_ℓ is isomorphic to a subgroup of $\mathrm{Aut}(K, L)$.

So let us consider an arbitrary element $\eta := (\varphi, \hat{\varphi}) \in X$, with parameters $(g, \hat{g})$. Recall that

$$\varphi(vw) = g\varphi(v)\hat{\varphi}(w) \tag{3.10}$$

$$\hat{\varphi}(wv) = \hat{g}\hat{\varphi}(w)\varphi(v) \tag{3.11}$$

for all $v \in V$ and all $w \in W$. If we set $w = \delta$ in (3.10), then we get that $\varphi(v) = g\varphi(v)\hat{\varphi}(\delta)$, and hence

$$\varphi(v)\hat{\varphi}(\delta) = g^{-1}\varphi(v) = \varphi(v) \cdot [g^{-1}] \ .$$

Since $[g^{-1}] \in \mathrm{Rad}(\hat{f})$, it follows from $(\mathbf{C}_{11})$ that $\varphi(v) \cdot (\hat{\varphi}(\delta) + [g^{-1}]) = 0$, and hence $\hat{\varphi}(\delta) = [g^{-1}]$.

If we apply $\hat{\varphi}$ on $(\mathbf{C}_{12})$, then it follows from (3.11) that

$$\hat{\varphi}([f(v_1 w, v_2)]) = \hat{g}[f(\varphi(v_1)\hat{\varphi}(w), \varphi(v_2))] \in [K]$$

for all $v_1, v_2 \in V$ and all $w \in W$. Since $\mathrm{Im}(f) = K$, it now follows that $\hat{\varphi}([K]) \subseteq [K]$. We can thus define a map $\rho = \rho_\eta : K \to K$ such that

$$\hat{\varphi}([t]) = [g^{-1}\rho(t)] \tag{3.12}$$

for all $t \in K$. In particular, $\rho(0) = 0$ and $\rho(1) = 1$.

If we now set $w = [t]$ in (3.10) for some $t \in K$, then we get that

$$\varphi(tv) = g\varphi(v)\hat{\varphi}([t]) = g\varphi(v)[g^{-1}\rho(t)] = \rho(t)\varphi(v) \tag{3.13}$$

for all $v \in V$. In particular, it follows from (3.13) that ρ is multiplicative; hence ρ is a field automorphism of K. Similarly, there is a field automorphism $\hat{\rho} = \hat{\rho}_\eta$ of L such that

$$\hat{\varphi}(sw) = \hat{\rho}(s)\hat{\varphi}(w) \tag{3.14}$$

for all $s \in L$ and all $w \in W$.

Now let $s \in L$ and $v \in V$ be arbitrary. By (3.13), we have that $\varphi(sv) = \rho(s)\varphi(v)$. On the other hand, it follows from Remark 2.7.97 that $sv = v \cdot s\delta$, and hence, by (3.10), (3.14) and Remark 2.7.97,

$$\begin{aligned}\varphi(sv) &= \varphi(v \cdot s\delta) = g\varphi(v)\hat{\varphi}(s\delta) = g\varphi(v) \cdot \hat{\rho}(s)\hat{\varphi}(\delta) \\ &= g\varphi(v) \cdot \hat{\rho}(s)[g^{-1}] = g\varphi(v) \cdot [\hat{\rho}(s)g^{-1}] = \hat{\rho}(s)\varphi(v)\ ,\end{aligned}$$

and it follows that $\rho(s) = \hat{\rho}(s)$, for all $s \in L$. Hence $\hat{\rho}$ is the restriction of ρ to L; in particular, $\rho(L) = L$, hence $\rho \in \mathrm{Aut}(K, L)$.

Now let Φ be the map from X to $\mathrm{Aut}(K, L)$ which maps every self-similarity $\eta = (\varphi, \hat{\varphi}) \in X$ to the corresponding $\rho_\eta \in \mathrm{Aut}(K, L)$. Then it follows from the fact that $\varphi(tv) = \rho_\eta(t)\varphi(v)$ for all $t \in K$ and all $v \in V$ that Φ is a group homomorphism. The kernel of Φ consists of the self-similarities η for which the corresponding field automorphisms ρ_η and hence also $\hat{\rho}_\eta$ are trivial – those are precisely the linear self-similarities. Hence $\mathrm{Ker}(\Phi) = X_\ell$, and it follows that

$$X/X_\ell \cong \mathrm{Im}(\Phi) \leq \mathrm{Aut}(K, L) \;, \tag{3.15}$$

which is what we wanted to show.

From now on, let us assume that $\eta = (\varphi, \hat{\varphi}) \in X_\ell$. Our goal is to show that $\eta \in X^\dagger$, and we will do this in several steps. In each case, we will multiply this given element by other elements in $X^\dagger$ to reduce to the next case.

3.4.2 Reduction to the Case $\hat{g} = 1$

Since $\eta \in X_\ell$, the map ρ is now the identity map, so it follows from (3.12) that

$$\hat{\varphi}([t]) = [g^{-1}t] \tag{3.16}$$

for all $t \in K$. Similarly, we have that $\varphi([s]) = [\hat{g}^{-1}s]$ for all $s \in L$; in particular, $\varphi(\epsilon) = [\hat{g}^{-1}]$.

If we set $w = \delta$ in (3.11), we get by $(\mathbf{C}_5)$ that $\hat{\varphi}([q(v)]) = \hat{g}[g^{-1}]\varphi(v)$, and hence, by (3.16) and $(\mathbf{C}_5)$, $[g^{-1}q(v)] = \hat{g}[g^{-1}q(\varphi(v))] = [\hat{g}g^{-1}q(\varphi(v))]$ (see Remark 2.7.97). We conclude that

$$q(\varphi(v)) = \hat{g}^{-1}q(v) \tag{3.17}$$

for all $v \in V$, and hence φ is a similitude of q with multiplier $\hat{g}^{-1}$. It now follows from Theorem 3.3.3 that $\hat{g}^{-1} = t_0^2\hat{q}(w_1)\hat{q}(w_2)$ for some $t_0 \in K^*$ and some $w_1, w_2 \in W^*$.

By Lemma 3.4.3 and Remark 3.2.2, $\zeta := \hat{\chi}_{[t_0]}^{-1}\hat{\chi}_{w_1}^{-1}\hat{\chi}_{w_2}^{-1}$ is a self-similarity of Ω with parameters $(1, t_0^2\hat{q}(w_1)\hat{q}(w_2)) = (1, \hat{g}^{-1})$. Moreover, $\zeta \in X^\dagger$. Again by Remark 3.2.2, $\eta\zeta$ is a linear self-similarity with parameters $(g, 1)$, and $\eta \in X^\dagger$ if and only if $\eta\zeta \in X^\dagger$.

We have thus reduced the problem to the case where $\hat{g} = 1$, and we will assume this from now on.

3.4.3 Reduction to the Case $\varphi = 1$

By (3.17), we now have that $q(\varphi(v)) = q(v)$ for all $v \in V$, that is, φ is an isometry of q. Moreover, $\varphi([s]) = [s]$ for all $s \in L$, that is, φ acts trivially on $[L]$. This allows us to prove a Dieudonné-Cartan-type theorem for φ, even though q is *not* regular and not necessarily finite dimensional. Note, however, that q *is* anisotropic.

Theorem 3.4.4. *φ is the product of at most 4 reflections in V.*

Proof. The proof is completely similar to the proof in the regular anisotropic case.

Let J denote the fixed point set of φ. Note that $[L] \subseteq J$, so by Theorems 2.7.99 and 2.7.100, J has codimension at most 4 in V. If $\varphi = 1$, then we are done, so we can assume that $J \neq V$; choose an arbitrary element $d \in V \setminus J$. Let $b := \varphi(d) + d \in V$; then $b \neq 0$. Then

$$q(d) = q(\varphi(d)) + q(b) + f(b, \varphi(d)) \ ,$$

which implies that $q(b) = f(b, \varphi(d))$. Hence

$$\pi_b\varphi(d) = \varphi(d) + f(\varphi(d), b)q(b)^{-1}b = \varphi(d) + b = d \ ,$$

so $\pi_b\varphi$ fixes the element d. On the other hand, if c is an arbitrary element of J, then

$$\pi_b\varphi(c) = \pi_b(c) = c + f(c, b)q(b)^{-1}b = c \ ,$$

since

$$f(c, b) = f(c, \varphi(d) + d) = f(\varphi(c), \varphi(d)) + f(c, d) = 0 \ .$$

Hence $\pi_b\varphi$ acts trivially on $\langle J, d\rangle$, since φ is K-linear. Since $d \notin J$, we have that $\operatorname{codim}_K\langle J, d\rangle = \operatorname{codim}_K J - 1$. By induction, it now follows that φ is the product of at most 4 reflections. $\square$

By Theorem 3.4.4, there exist four elements $c_1, c_2, c_3, c_4 \in V$ such that $\varphi = \pi_{c_1}\pi_{c_2}\pi_{c_3}\pi_{c_4}$ (note that $\pi_\epsilon = 1$). On the other hand,

$$\zeta' := \chi_{c_4}\chi_{c_3}\chi_{c_2}\chi_{c_1} : (v, w) \mapsto (\pi_{c_4}\pi_{c_3}\pi_{c_2}\pi_{c_1}(v), wc_1c_2c_3c_4) \ .$$

Again, note that $\zeta' \in X^\dagger$, and hence $\eta \in X^\dagger$ if and only if $\zeta'\eta \in X^\dagger$. Since

$$\zeta'\eta : (v, w) \mapsto (v, \hat{\varphi}(w)c_1c_2c_3c_4) \ ,$$

we have reduced the problem to the case where $\varphi = 1$, and we will assume this from now on.

3.4.4 Determination of $\hat{\varphi}$

Our next goal is to show that $g \in L^*$. We start with a lemma.

Lemma 3.4.5. *Let $\xi \in W \setminus [K]$ and $z \in W$ be arbitrary elements such that $\hat{f}(\xi v, z) = 0$ for all $v \in V$. Then there exist an element $s \in L$ and an element $t \in K$ such that $z = s\xi + [t]$.*

Proof. Let $d \in V \setminus [L]$ be arbitrary, and let e be as in Theorem 2.7.98. By Theorem 2.7.101, $z = s_1\xi + s_2\xi ed^{-1} + s_3\xi d^{-1} + s_4\beta^2\xi e + [t]$ for some $s_1, s_2, s_3, s_4 \in L$ and some $t \in K$. It follows from $\hat{f}(\xi, z) = 0$, $\hat{f}(\xi d^{-1}, z) = 0$ and $\hat{f}(\xi e, z) = 0$ that $s_2 = 0$, $s_4 = 0$ and $s_3 = 0$, respectively. We conclude that $z = s_1\xi + [t]$, which is what we had to show. □

We now pick up our examination of $\eta = (1, \hat{\varphi}) \in X_\ell$. Let $w \in W \setminus [K]$ be arbitrary, and let $z := \hat{\varphi}(w)$. Then it follows from (3.10) that

$$vw = gvz \tag{3.18}$$

for all $v \in V$. If we apply q on both sides of (3.18), then it follows from Lemmas 2.7.95(ii) and 2.7.97(i) that

$$\hat{q}(w) = g^2\hat{q}(z) \ . \tag{3.19}$$

Lemma 3.4.6. $\hat{f}(w, z) = 0$.

Proof. Let $v \in V \setminus [L]$ be arbitrary. By Lemma 3.4.1(ii), (3.18), (3.19), ($\mathbf{C}_7$) and Remark 2.7.97,

$$\begin{aligned} v\hat{\pi}_z(w) &= vz^{-1}wz = \hat{q}(z)^{-1}vzwz = \hat{q}(z)^{-1}g^{-1}vwwz \\ &= \hat{q}(z)^{-1}g^{-1}\hat{q}(w)vz = \hat{q}(z)^{-1}g^{-2}\hat{q}(w)vw = vw \ , \end{aligned}$$

and hence, by ($\mathbf{C}_{11}$),

$$\begin{aligned} [\hat{f}(wv, \hat{\pi}_z(w))] &= v(w + \hat{\pi}_z(w)) + vw + v\hat{\pi}_z(w) \\ &= v \cdot \hat{f}(w, z)\hat{q}(z)^{-1}z \ , \end{aligned}$$

which implies that $v \cdot \hat{f}(w, z)\hat{q}(z)^{-1}z \in [L]$. Since $v \notin [L]$, it follows that $\hat{f}(w, z) = 0$. □

Lemma 3.4.7. $\hat{f}(wv, z) = 0$ *for all* $v \in V$.

Proof. By Lemma 3.4.6, $\hat{\pi}_z(w) = w$. It follows from Lemma 3.4.1(iv) that

$$wv + \hat{f}(wv, z)\hat{q}(z)^{-1}z = \hat{q}(z)^{-1}w \cdot vz .$$

By (3.18), Remark 2.7.97, (3.19) and ($\mathbf{C}_{10}$),

$$\hat{q}(z)^{-1}w \cdot vz = \hat{q}(z)^{-1}g^{-2}w \cdot vw = \hat{q}(w)^{-1}\hat{q}(w)wv = wv ,$$

and it follows that $\hat{f}(wv, z)\hat{q}(z)^{-1}z = 0$, which implies that $\hat{f}(wv, z) = 0$. □

By Lemma 3.4.7, we are now ready to invoke Lemma 3.4.5. It follows that $z = sw + [t]$ for some $s \in L$ and some $t \in K$. If we plug in this expression for z in (3.18), then we get that

$$vw = gv(sw + [t]) = (sg)vw + (tg)v .$$

But since we chose $w \notin [K]$, the elements v and vw are linearly independent (for if there were an $x \in K$ such that $vw = xv$, then $v(w + [x]) = 0$ by ($\mathbf{C}_{11}$) and hence $w = [x] \in [K]$). It thus follows that $tg = 0$ and $sg = 1$, hence s is invertible, $g = s^{-1} \in L^*$, and $z = g^{-1}w$.

Since $w \in W \setminus [K]$ was arbitrary, it follows that $\hat{\varphi}(w) = g^{-1}w$ for all $w \in W \setminus [K]$. Moreover, since $g^{-1} \in L$, it follows from (3.16) and Remark 2.7.97 that $\hat{\varphi}([t]) = g^{-1}[t]$ for all $w = [t] \in [K]$ as well. So $\hat{\varphi}$ is just scalar multiplication by the element $g^{-1} \in L^*$.

It now suffices to observe that $\pi_{[g^{-1}]} = 1$ to conclude that $\eta = \chi_{[g^{-1}]}$. Since $\chi_{[g^{-1}]} \in X^\dagger$, we have shown that $X_\ell = X^\dagger$.

It finally follows from (1.1), (3.9) and (3.15) that

$$G/G^\dagger \cong H/H^\dagger \cong X/X^\dagger = X/X_\ell \cong \mathrm{Im}(\Phi) \leq \mathrm{Aut}(K, L) ,$$

which proves the Main Theorem 3.1.1.

4 Quadratic Forms of Type E_6, E_7 and E_8

In Theorem 2.7.14 of Chapter 2, we have shown that a wide quadrangular system Ω contains a canonically embedded reduced sub-quadrangular system $\Gamma = \Gamma_\Omega$. In one case, Γ is the quadrangular system determined by an anisotropic quadratic space (K, V_0, q); this case has been examined in Theorem 2.7.10. One of the results of this theorem says that if q is regular and the dimension of V_0 is greater than four, then q must be a quadratic form of type E_k for $k = 6$, 7 or 8 as defined on page 64 and Ω is uniquely determined by Γ and hence by q.

By the results of [43] (see the appendix of [52] for details), on the other hand, to each anisotropic quadratic form q over a field K whose even Clifford algebra has the structure given in 4.3.4 below, a K-form of a simple algebraic group of type E_k exists whose spherical building is a wide Moufang quadrangle Ω such that Γ_Ω is the quadrangle determined by q. These results imply that the only quadratic forms whose even Clifford algebras are as in 4.3.4 are quadratic forms of type E_k (for $k = 6$, 7 or 8). In this chapter, we give a direct proof of this result.

During the proof of this fact, we also obtain new general results about low-dimensional quadratic forms.

4.1 Some Basic Properties about Even Dimensional Quadratic Forms

We start this chapter by mentioning some basic lemmas and theorems about even dimensional quadratic forms which we will need later on.

Lemma 4.1.1. *Let E/K be a separable quadratic extension with norm N, and let $\gamma \in K^*$. Then $(E/K, \gamma\tilde{d}(N)) \cong (E/K, -\gamma)$, where $\tilde{d}(q) \in K^*/(K^*)^2$*

is defined as

$$\tilde{d}(q) = \begin{cases} d(q) & \text{if } \mathrm{char}(K) \text{ is odd }, \\ 1 & \text{if } \mathrm{char}(K) \text{ is even }. \end{cases}$$

Proof. By Theorem 1.7.3, we have to prove that

$$[E/K, \gamma\tilde{d}(N)] + [E/K, -\gamma] = 0 \,,$$

which is in turn equivalent to $[E/K, -\gamma^2\tilde{d}(N)] = 0$, by Theorem 1.7.1. By Lemma 1.7.2, this is equivalent to $-\tilde{d}(N) \in N(E)$. If $\mathrm{char}(K) = 2$, then this is a trivial statement; if $\mathrm{char}(K) \neq 2$, then this follows from the fact that N (considered as a quadratic form) has a diagonalization $N \simeq \langle 1, -d(N)\rangle$. □

Lemma 4.1.2. *Let E/K be a separable quadratic extension with norm N, and let $s \in K^*$. Then $C(sN) \cong (E/K, s)$.*

Proof. See, for example, [52, (12.27)] for a characteristic-free proof of this well known fact. □

Lemma 4.1.3. *If q and q' are two even dimensional regular quadratic forms over K, then $c(q \perp q') = c(q) + c(\tilde{d}(q)q')$ in $\mathrm{Br}(K)$, where $\tilde{d}(q)$ is defined as in Lemma 4.1.1.*

Proof. If $\mathrm{char}(K) \neq 2$, then this can be found in [29, V.3.15]. If $\mathrm{char}(K) = 2$, then the graded tensor product is the same as the ordinary tensor product, so

$$C(q \perp q') \cong C(q) \mathbin{\widehat{\otimes}_K} C(q') \cong C(q) \otimes_K C(q') \,,$$

(see, for example, [41, 9.2.5] for a characteristic-free proof of the first isomorphism) and hence $c(q \perp q') = c(q) + c(q')$. □

Theorem 4.1.4. (See [52, (12.28)].) *Let E/K be a separable quadratic extension with norm N, and let $s_1, \dots, s_d \in K^*$. Then*

$$c(s_1 N \perp \cdots \perp s_d N) = [E/K, (-1)^{[d/2]} s_1 \dots s_d] \,.$$

Proof. This follows by induction on d, using Lemmas 4.1.2, 4.1.3 and 4.1.1. □

Lemma 4.1.5. *If q and q' are two even dimensional regular quadratic forms over K, then*

$$d(q \perp q') = \begin{cases} d(q)d(q') & \text{if } \mathrm{char}(K) \neq 2 \,, \\ d(q) + d(q') & \text{if } \mathrm{char}(K) = 2 \,. \end{cases}$$

In both cases, $d(q \perp q')$ is trivial if and only if $d(q) = d(q')$.

Proof. First assume that $\mathrm{char}(K) \neq 2$. If q is $2d$-dimensional, then

$$d(q) = (-1)^{2d(2d-1)/2}\det(q) = (-1)^d \det(q) \ ,$$

so d is clearly multiplicative on even dimensional quadratic forms.

If $\mathrm{char}(K) = 2$, then it follows immediately from the definition that $d(q \perp q') = d(q) + d(q')$. □

Theorem 4.1.6. (See [52, (12.43)].)

(i) *Let q be a 6-dimensional regular anisotropic quadratic form over K.* **If** *q is of type E_6,* **then** *$d(q)$ is non-trivial and $c(q) = [Q]$ for some quaternion algebra Q over K which, if it is a division algebra, contains the discriminant extension of q.*

(ii) *Let q be an 8-dimensional regular anisotropic quadratic form over K.* **If** *q is of type E_7,* **then** *$d(q)$ is trivial and $c(q) = [D]$ for some quaternion division algebra D over K.*

(iii) *Let q be a 12-dimensional regular anisotropic quadratic form over K.* **If** *q is of type E_8,* **then** *$d(q)$ is trivial and $c(q) = 0$.*

Proof. First of all, we observe that, if q is an anisotropic $2d$-dimensional quadratic form over K with a norm splitting $q \simeq s_1 N \perp \cdots \perp s_d N$, where N is the norm of some separable quadratic extension E/K and $s_1, \ldots, s_d \in K^*$, then it follows from Lemma 4.1.5 that

$$d(q) = \begin{cases} d(N) & \text{if } d \text{ is odd} \ , \\ \text{trivial} & \text{if } d \text{ is even} \ . \end{cases}$$

The statements about $d(q)$ now follow from the fact that $d(N)$ is non-trivial. A straightforward application of Theorem 4.1.4 together with Lemma 1.7.2 yields the statements about $c(q)$. □

The goal of the next section is to prove the converse of this theorem.

4.2 Characterizations

Now that we have assembled the necessary tools, we can start to prove some characterizations for even dimensional quadratic forms. We will gradually increase the dimension. But first of all, we start with three lemmas which we will make use of repeatedly.

Lemma 4.2.1. *Let E/K be a separable quadratic extension with norm N. If (V, q) is an anisotropic regular quadratic space such that $(V \otimes_K E, q_E)$ is isotropic, then $q \simeq sN \perp q'$ for some $s \in K^*$ and some regular quadratic form q' over K.*

Proof. If $\mathrm{char}(K) \neq 2$, then this can be found in [41, 2.5.1]. The proof in the case $\mathrm{char}(K) = 2$ is similar, as we will show. Let $E = K(\alpha)$ where $\alpha^2 + \alpha + \delta = 0$, with $\delta = d(N)$. Then every element $w \in V \otimes_K E$ can be written (in a unique way) as $w = x + y\alpha$ for some $x, y \in V$. Let $u, v \in V$ (not both trivial) be such that $u + v\alpha$ is isotropic. Since (V, q) is anisotropic, we must have $v \neq 0$. Then

$$\begin{aligned} 0 &= q(u + v\alpha) \\ &= q(u) + \alpha^2 q(v) + \alpha f(u, v) \\ &= q(u) + \delta q(v) + \alpha(q(v) + f(u, v)) , \end{aligned}$$

which implies that $q(u) = \delta q(v)$ and $f(u, v) = q(v)$. If we set $s = q(v) \in K^*$, then the matrix corresponding to the restriction of q to the subspace with ordered basis (u, v), is given by $\left(\begin{smallmatrix} \delta s & s \\ 0 & s \end{smallmatrix}\right) = s\left(\begin{smallmatrix} \delta & 1 \\ 0 & 1 \end{smallmatrix}\right)$. But this is exactly the matrix representation of sN as a quadratic form; see, for example, [41, 9.4.1]. So the restriction of q to the subspace generated by u and v in V is isometric to sN. Since $\langle u, v \rangle$ is regular, $V \cong \langle u, v \rangle \perp \langle u, v \rangle^{\perp}$, and hence $q \simeq sN \perp q'$ for some regular quadratic form q' over K. $\square$

Lemma 4.2.2. *Let E/K be a separable quadratic extension with norm N. If (V, q) is an anisotropic regular quadratic space such that $(V \otimes_K E, q_E)$ is hyperbolic, then q has a norm splitting $q \simeq s_1 N \perp \cdots \perp s_d N$ with $s_1, \ldots, s_n \in K^*$.*

Proof. (See [41, 2.5.11]). Since q_E is isotropic, $q \simeq s_1 N \perp q'$ for some scalar $s_1 \in K^*$ and some regular quadratic form q', by Lemma 4.2.1. Since q_E is hyperbolic, so is q'_E. By induction on the dimension, we get that q has a norm splitting $q \simeq s_1 N \perp \cdots \perp s_d N$ with $s_1, \ldots, s_n \in K^*$. $\square$

The following key lemma will allow us to decrease the dimension by tensoring up by some quadratic extension.

Lemma 4.2.3. *Let q be an even dimensional regular quadratic form over K, with a decomposition $q \simeq q_1 \perp q'$, where q_1 is a 2-dimensional regular quadratic form.*

(i) *Suppose that $d(q_1)$ is trivial. Then $d(q') = d(q)$ and $c(q') = c(q_1) + c(q)$.*

(ii) *Suppose that $d(q_1)$ is non-trivial. Consider the discriminant extension E/K of q_1. Then $d(q'_E) = d(q_E)$ and $c(q'_E) = c(q_E)$.*

Proof. (i) Since $d(q_1)$ is trivial, it follows from Lemma 4.1.5 that $d(q) = d(q_1 \perp q') = d(q')$. Since $\tilde{d}(q_1) = 1$, Lemma 4.1.3 implies that $c(q) = c(q_1 \perp q') = c(q_1) + c(q')$. The result follows, since $c(q_1)$ is of order at most 2 in $\mathrm{Br}(K)$.

(ii) Since E/K is the discriminant extension of q_1, we have that $d((q_1)_E)$ is trivial. Denote the norm of E/K by N. Then $q_1 \simeq sN$ for some $s \in K^*$. It follows that

$$c((q_1)_E) = [C(sN) \otimes_K E] = [(E/K, s) \otimes_K E] = 0$$

in $\mathrm{Br}(E)$ by Lemma 4.1.2 and Lemma 1.6.6(ii). It now follows from part (i) that $d(q'_E) = d(q_E)$ and $c(q'_E) = c((q_1)_E) + c(q_E) = c(q_E)$. □

We can now start with some characterizations of 4-dimensional quadratic forms.

Lemma 4.2.4. *A* 4*-dimensional regular quadratic form q over K with trivial discriminant has a norm splitting or is hyperbolic.*

Proof. Let $q \simeq q_1 \perp q_2$, where q_1 and q_2 are 2-dimensional regular quadratic forms. Since $d(q_1 \perp q_2)$ is trivial, it follows immediately from Lemma 4.1.5 that $d(q_1) = d(q_2)$. If this discriminant is trivial, then q is hyperbolic; if this discriminant is non-trivial, then q has a norm splitting. □

The following lemma is well known; see, for example, [41, 2.14.3.iii] for a proof in odd characteristic. Our proof is characteristic-free.

Lemma 4.2.5. *A* 4*-dimensional regular quadratic form q over K with trivial discriminant and $c(q) = 0$ is hyperbolic.*

Proof. Let (V, q) be the quadratic space corresponding to q. By Lemma 4.2.4, q is hyperbolic or q has a norm splitting. If q is hyperbolic, then we are done, so we can assume that q has a norm splitting. Then there exists a separable quadratic extension E/K with norm N and two constants $s_1, s_2 \in K^*$ such that $q \simeq s_1 N \perp s_2 N$. By Theorem 4.1.4, $c(q) = [E/K, -s_1 s_2]$. Since $c(q) = 0$, this implies that $-s_1 s_2 \in N(E)$, by Lemma 1.7.2. So there exists a $t \in E^*$ such that $s_2 = -s_1 N(t)$, and hence

$$q \simeq s_1 N \perp -s_1 N(t) N \, .$$

The subspace consisting of the elements of V corresponding to the elements $(tx, x) \in E \oplus E$ for all $x \in E$, is 2-dimensional and totally isotropic. Thus q is hyperbolic. □

The next lemma was first proved by Albert (see [1]), but his proof involves a lot of calculation, and only works in odd characteristic. A shorter proof in odd characteristic can be found in [41, 2.14.3.iv]. A characteristic-free but very indirect proof can be found in [28, 16.5]. We will give a direct characteristic-free proof.

Theorem 4.2.6. *A 6-dimensional regular quadratic form q over K with trivial discriminant and $c(q) = 0$ is hyperbolic.*

Proof. Consider a decomposition $q \simeq q_1 \perp q_2 \perp q_3$, where each q_i is a 2-dimensional regular quadratic form. If $d(q_1) = d(q_2) = d(q_3)$ are all trivial, then q is hyperbolic, and we are done. So without loss of generality, we can assume that $d(q_1)$ is not trivial. Consider its discriminant extension E/K with norm N. In particular, $q_1 \simeq sN$ for some $s \in K^*$. Let $q' = q_2 \perp q_3$, then $q \simeq q_1 \perp q'$. We know that $d(q_E)$ is trivial and that $c(q_E) = 0$. By the Key Lemma 4.2.3(ii), $d(q'_E) = d(q_E)$, so $d(q'_E)$ is trivial as well, and $c(q'_E) = c(q_E) = 0$. By Lemma 4.2.5, q'_E is hyperbolic.

Suppose that q' is anisotropic. Since q'_E is hyperbolic, it then follows from Lemma 4.2.2 that q' has a norm splitting $q' \simeq s_2N \perp s_3N$ for some $s_2, s_3 \in K^*$. So $q \simeq q_1 \perp q' \simeq sN \perp s_2N \perp s_3N$. It follows from Lemma 4.1.5 that $d(q) = d(N)$, but $d(q)$ is trivial, whereas $d(N)$ is not trivial – a clear contradiction. So q' has to be isotropic, hence q is isotropic as well.

Thus q has a decomposition $q \simeq h \perp \tilde{q}$, where h is a 2-dimensional hyperbolic quadratic form. Since $d(h)$ is trivial and $c(h) = 0$, the Key Lemma 4.2.3(i) tells us that $d(\tilde{q})$ is trivial and $c(\tilde{q}) = 0$. Again by Lemma 4.2.5, this implies that $\tilde{q}$ is hyperbolic, and so q is hyperbolic as well. □

Lemma 4.2.7. *Let q be a 6-dimensional anisotropic regular quadratic form over K with $d(q)$ non-trivial. Consider the discriminant extension E/K of q. If $c(q_E) = 0$ in $\mathrm{Br}(E)$, then q has a norm splitting.*

Proof. We tensor everything up by E. Then q_E is a 6-dimensional regular quadratic form with trivial discriminant (since E is exactly the discriminant extension of q) and with $c(q_E) = 0$. By Theorem 4.2.6, this implies that q_E is hyperbolic. Since q is anisotropic, Lemma 4.2.2 implies that q has a norm splitting. □

An odd-characteristic version of the following lemma can be found in [41, 2.14.3.v].

Lemma 4.2.8. *A 6-dimensional regular quadratic form q over K with trivial discriminant and $c(q) = [Q]$ for some quaternion algebra Q over K, is isotropic.*

Proof. If Q is not a division algebra, then $c(q) = 0$, and the result follows from Theorem 4.2.6. So we can assume that Q is a quaternion division algebra. In particular, Q contains a subfield E such that E/K is a separable quadratic extension; we denote its norm by N. If we tensor everything up by this extension field E, then we get a quadratic form q_E over E with trivial discriminant and with $c(q_E) = [Q \otimes_K E] = 0$ in $\mathrm{Br}(E)$, by Lemma 1.6.6(ii). Again by Theorem 4.2.6, this implies that q_E is hyperbolic.

Suppose that q were anisotropic. Then Lemma 4.2.2 would imply that q has a norm splitting $q \simeq s_1 N \perp s_2 N \perp s_3 N$ for some $s_1, s_2, s_3 \in K^*$. By Lemma 4.1.5, it would follow from this that $d(q) = d(N)$, which contradicts the fact that q has trivial discriminant. We conclude that q must be isotropic. □

Theorem 4.2.9. *An 8-dimensional anisotropic regular quadratic form q over K with trivial discriminant and $c(q) = [Q]$ for some quaternion algebra Q over K, has a norm splitting.*

Proof. Consider a decomposition $q \simeq q_1 \perp q_2 \perp q_3 \perp q_4$, where each q_i is a 2-dimensional regular quadratic form, and set $q' = q_2 \perp q_3 \perp q_4$. Since q is anisotropic, so is q_1, hence $d(q_1)$ is non-trivial, and we can consider its discriminant extension E/K with norm N. In particular, $q_1 \simeq s_1 N$ for some $s_1 \in K^*$.

If we tensor everything up by this extension field E, then q_E still has trivial discriminant, and $c(q_E) = [Q \otimes_K E]$ in $\mathrm{Br}(E)$. Note that $Q \otimes_K E$ is a quaternion algebra over E. It now follows from the Key Lemma 4.2.3(ii) that $d(q'_E) = d(q_E)$, so $d(q'_E)$ is trivial, and $c(q'_E) = c(q_E) = [Q \otimes_K E]$. Hence we can apply Lemma 4.2.8 on q'_E, and we get that q'_E is isotropic. Since q' is anisotropic, this implies by Lemma 4.2.1 that $q' \simeq s_2 N \perp q''$ for some $s_2 \in K^*$ and some regular 4-dimensional quadratic form q''.

So $q \simeq s_1 N \perp s_2 N \perp q''$. Since $d(q)$ is trivial, Lemma 4.1.5 implies that $d(q'') = d(s_1 N \perp s_2 N)$. Since $d(s_1 N) = d(s_2 N)$, this same lemma also implies that $d(s_1 N \perp s_2 N)$ is trivial, and hence $d(q'')$ is trivial as well. Note that q'' is anisotropic, since q is anisotropic. So by Lemma 4.2.4, q'' has a norm splitting $q'' \simeq s_3 N' \perp s_4 N'$, where N' is the norm of some separable quadratic extension E'/K, and $s_3, s_4 \in K^*$.

So $q \simeq s_1 N \perp s_2 N \perp s_3 N' \perp s_4 N'$. Since $d(s_1 N \perp s_2 N)$ is trivial, Lemma 4.1.3 implies that $c(q) = c(s_1 N \perp s_2 N) + c(s_3 N' \perp s_4 N')$. By Theorem 4.1.4, this implies that $c(q) = [E/K, -s_1 s_2] + [E'/K, -s_3 s_4]$. But $c(q) = [Q]$ as well, so by Theorem 1.7.4, there exists a separable quadratic extension $\tilde{E}/K$ with norm $\tilde{N}$ which splits all three quaternion algebras $(E/K, -s_1 s_2)$, $(E'/K, -s_3 s_4)$ and Q. In particular,

$$c((s_1 N \perp s_2 N)_{\tilde{E}}) = [(E/K, -s_1 s_2) \otimes_K \tilde{E}] = 0 \in \mathrm{Br}(\tilde{E}) \ ,$$

and of course, $d((s_1N \perp s_2N)_{\tilde{E}})$ is trivial as well. So by Lemma 4.2.5, $(s_1N \perp s_2N)_{\tilde{E}}$ is hyperbolic. Since $s_1N \perp s_2N$ is anisotropic, it follows from Lemma 4.2.2 that $s_1N \perp s_2N$ has a norm splitting $s_1N \perp s_2N \simeq t_1\tilde{N} \perp t_2\tilde{N}$ for some $t_1, t_2 \in K^*$. A completely similar argument is true for $s_3N' \perp s_4N'$, hence $s_3N' \perp s_4N' \simeq t_3\tilde{N} \perp t_4\tilde{N}$ for some $t_3, t_4 \in K^*$.

We conclude that $q \simeq t_1\tilde{N} \perp t_2\tilde{N} \perp t_3\tilde{N} \perp t_4\tilde{N}$. So q has a norm splitting. □

Theorem 4.2.10. *A* 10*-dimensional regular quadratic form q over K with trivial discriminant and $c(q) = 0$ is isotropic.*

Proof. We follow a part of the proof of [41, 2.14.4], which is in odd characteristic only, but which can be extended to the general case without any significant change.

Consider a decomposition $q \simeq q_4 \perp q_6$, where q_4 and q_6 are regular quadratic forms of dimension 4 and 6 respectively. By Lemma 4.1.5, $d(q_4) = d(q_6)$.

First, assume that $d(q_4) = d(q_6)$ is trivial. Then Lemma 4.2.4 and Theorem 4.1.4 together imply that $c(q_4) = [Q]$ for some quaternion algebra Q over K. Since $\tilde{d}(q_4) = 1$, Lemma 4.1.3 implies that $c(q_6) = c(q_4) = [Q]$. By Lemma 4.2.8, q_6 is isotropic, and so is q.

So we can assume that $d(q_4) = d(q_6)$ is non-trivial. Consider the discriminant extension E/K of q_4 (or q_6) with norm N. Once again, we tensor everything up by this extension field E, and we get that $d((q_4)_E) = d((q_6)_E)$ is trivial. In the same way as in the previous paragraph, we obtain that $(q_6)_E$ is isotropic.

If q_6 is isotropic, then q is isotropic as well, and we are done; so we can assume that q_6 is anisotropic. Then Lemma 4.2.1 implies that q_6 has a decomposition $q_6 \simeq sN \perp q'$ for some $s \in K^*$ and some 4-dimensional regular quadratic form q'. Since $d(q_6) = d(N) = d(sN)$ (because N is exactly the norm of the discriminant extension of q_6), Lemma 4.1.5 implies that $d(q')$ is trivial, and Lemma 4.2.4 and Theorem 4.1.4 together imply that $c(q') = [Q']$ for some quaternion algebra Q' over K.

Now, consider a decomposition $q \simeq q' \perp \tilde{q}$, where $\tilde{q}$ is a 6-dimensional regular quadratic form. Since $d(q)$ and $d(q')$ are both trivial, so is $d(\tilde{q})$, by Lemma 4.1.5. Hence it follows from Lemma 4.1.3 that $c(\tilde{q}) = c(q') = [Q']$. Lemma 4.2.8 now implies that $\tilde{q}$ is isotropic, and we are done. □

Theorem 4.2.11. *A* 12*-dimensional anisotropic regular quadratic form q over K with trivial discriminant and $c(q) = 0$ has a norm splitting.*

Proof. Consider a decomposition $q \simeq q_2 \perp q_{10}$, where q_2 and q_{10} are regular quadratic forms of dimension 2 and 10 respectively. By Lemma 4.1.5,

$d(q_2) = d(q_{10})$. Since q_2 is anisotropic, this discriminant is non-trivial, and we can consider the discriminant extension E/K of q_2 (or q_{10}) with norm N. In particular, $q_2 \simeq s_1 N$ for some $s_1 \in K^*$.

Since $d(q)$ is trivial and $c(q) = 0$, the Key Lemma 4.2.3(ii) implies that $d((q_{10})_E)$ is trivial and $c((q_{10})_E) = 0$. Theorem 4.2.10 now implies that $(q_{10})_E$ is isotropic, so it follows from Lemma 4.2.1 that q_{10} has a decomposition $q_{10} \simeq s_2 N \perp q_8$, for some regular 8-dimensional quadratic form q_8.

So $q \simeq s_1 N \perp s_2 N \perp q_8$. By Theorem 4.1.4, $c(s_1 N \perp s_2 N) = [Q]$ for some quaternion algebra Q. Since $\tilde{d}(s_1 N \perp s_2 N) = 1$, Lemma 4.1.3 implies that $c(q_8) = c(s_1 N \perp s_2 N) = [Q]$. Furthermore, it follows from Lemma 4.1.5 that $d(q_8) = d(s_1 N \perp s_2 N)$, hence $d(q_8)$ is trivial. Since q is anisotropic, so is q_8, and Theorem 4.2.9 now implies that q_8 has a norm splitting $q_8 \simeq s_3 N' \perp s_4 N' \perp s_5 N' \perp s_6 N'$, where N' is the norm of a separable quadratic extension E'/K and $s_3, s_4, s_5, s_6 \in K^*$.

So we already have $q \simeq s_1 N \perp s_2 N \perp s_3 N' \perp s_4 N' \perp s_5 N' \perp s_6 N'$. Since $d(s_1 N \perp s_2 N)$ is trivial, Lemma 4.1.3 implies that

$$c(q) = c(s_1 N \perp s_2 N) + c(s_3 N' \perp s_4 N' \perp s_5 N' \perp s_6 N') \ .$$

By Theorem 4.1.4, this implies that $c(q) = [E/K, -s_1 s_2] + [E'/K, s_3 s_4 s_5 s_6]$, and hence $(E/K, -s_1 s_2) \cong (E'/K, s_3 s_4 s_5 s_6)$. It follows from Lemma 1.6.6(ii) that $[(E/K, -s_1 s_2) \otimes_K E'] = 0$ in $\mathrm{Br}(E')$, and hence $c((s_1 N \perp s_2 N)_{E'}) = 0$. Since $d((s_1 N \perp s_2 N)_{E'})$ is trivial as well, we have that $(s_1 N \perp s_2 N)_{E'}$ is hyperbolic, by Lemma 4.2.5. It now follows from Lemma 4.2.2 that $s_1 N \perp s_2 N$ has a decomposition $s_1 N \perp s_2 N \simeq t_1 N' \perp t_2 N'$ for some $t_1, t_2 \in K^*$.

We conclude that $q \simeq t_1 N' \perp t_2 N' \perp s_3 N' \perp s_4 N' \perp s_5 N' \perp s_6 N'$, so q has a norm splitting. □

We are now ready to prove the main theorem.

Theorem 4.2.12.

(i) *Let q be a 6-dimensional regular anisotropic quadratic form over K.* **If** *$d(q)$ is non-trivial and $c(q) = [Q]$ for some quaternion algebra Q over K which, if it is a division algebra, contains the discriminant extension of q,* **then** *q is of type E_6.*

(ii) *Let q be an 8-dimensional regular anisotropic quadratic form over K.* **If** *$d(q)$ is trivial and $c(q) = [D]$ for some quaternion division algebra D over K,* **then** *q is of type E_7.*

(iii) *Let q be a 12-dimensional regular anisotropic quadratic form over K.* **If** *$d(q)$ is trivial and $c(q) = 0$,* **then** *q is of type E_8.*

Proof. (i) Let E/K be the discriminant extension of q. By Lemma 1.6.6(ii), $c(q_E) = [Q \otimes_K E] = 0$ in $\mathrm{Br}(E)$. By Lemma 4.2.7, this implies that q has a norm splitting, i.e. q is of type E_6.

(ii) By Theorem 4.2.9, q has a norm splitting $q \simeq s_1N \perp \cdots \perp s_4N$ for some separable quadratic extension E/K with norm N and some $s_1, s_2, s_3, s_4 \in K^*$. By Theorem 4.1.4, $c(q) = [E/K, s_1s_2s_3s_4]$. Since $c(q) = [D]$, $[E/K, s_1s_2s_3s_4] \neq 0$ in $\mathrm{Br}(K)$, and Lemma 1.7.2 now implies that $s_1s_2s_3s_4 \notin N(E)$. Hence q is of type E_7.

(iii) By Theorem 4.2.11, q has a norm splitting $q \simeq s_1N \perp \cdots \perp s_6N$ for some separable quadratic extension E/K with norm N and some $s_1, \ldots, s_6 \in K^*$. By Theorem 4.1.4, $c(q) = [E/K, -s_1 \cdots s_6]$. Since $c(q) = 0$, Lemma 1.7.2 now implies that $-s_1 \cdots s_6 \in N(E)$. Hence q is of type E_8.

□

4.3 Even Clifford Algebra

We will now restate Theorems 4.1.6 and 4.2.12 in terms of the even Clifford algebra of q. The following relation between the Clifford algebra $C(q)$ and the even Clifford algebra $C_0(q)$ is well known. One possible reference is [20], where the whole theory is built up in a very general setup. Its statement of this theorem [20, 11.2] uses the polynomial of special elements, but this is equivalent to our discriminant assumptions. Namely, we can use [20, ex. 2, p. 101] to see that these assumptions correspond in the 2-dimensional case. Using an inductive argument and [20, Theorem 7.9, p. 102], we can extend this to the general even dimensional case.

Theorem 4.3.1. *Assume that q is an even dimensional regular quadratic form over K. Then there is a finite dimensional central division algebra D over K such that $C(q) \cong \mathrm{Mat}_k(D)$ as K-algebras, where both $\dim_K D$ and k are powers of 2. If $k > 1$, set $k = 2m$.*

(i) *If the discriminant $d(q)$ is trivial, then $C_0(q) \cong \mathrm{Mat}_m(D) \oplus \mathrm{Mat}_m(D)$, and $Z(C_0(q)) \cong K \oplus K$.*

(ii) *If the discriminant $d(q)$ is non-trivial, then we consider the discriminant extension E/K.*

  (a) *If E is contained in D, then $C_0(q) \cong \mathrm{Mat}_k(D_0)$ for some central division algebra $D_0 \subseteq D$ over E.*

  (b) *If E is not contained in D, then $D \otimes_K E$ is a central division algebra over E (see Lemma 1.6.6(i)), and $C_0(q) \cong \mathrm{Mat}_m(D \otimes_K E)$.*

In both cases (a) and (b), $Z(C_0(q)) \cong E$.

Proof. See [20, 11.2]. $\square$

Remark 4.3.2. A similar theorem exists for regular quadratic forms of odd dimension. See, for example, [20, 11.1].

We continue with a lemma that allows us to drop the regularity condition, because the regularity follows from the structure of the (even) Clifford algebra.

Lemma 4.3.3.

(i) *If* q_1 *and* q_2 *are two anisotropic quadratic forms over* K *of the same dimension and* $C_0(q_1) \cong C_0(q_2)$*, then* q_1 *and* q_2 *are both regular or both non-regular.*

(ii) *If* q_1 *and* q_2 *are two anisotropic quadratic forms over* K *of the same dimension and* $C(q_1) \cong C(q_2)$*, then* q_1 *and* q_2 *are both regular or both non-regular.*

Proof. Let q_1 and q_2 be two anisotropic quadratic forms over K of the same dimension, and suppose that either $C_0(q_1) \cong C_0(q_2)$ or $C(q_1) \cong C(q_2)$ (or both). We can assume that both q_1 and q_2 are quadratic forms from the same vector space V (over K) to K. If $\mathrm{char}(K) \neq 2$, then there is nothing to prove, since all anisotropic quadratic forms are regular in this case. So we can assume that $\mathrm{char}(K) = 2$. If $\dim_K V$ is odd, then there is again nothing to prove, because a regular quadratic form in even characteristic is always even dimensional. So we can also assume that $\dim_K V$ is even.

Suppose that q_1 is regular and that q_2 is non-regular. By the structure Theorem 4.3.1, we know that $Z(C(q_1)) = K$, and either $Z(C_0(q_1)) \cong K \oplus K$ or $Z(C_0(q_1)) \cong E$ where E/K is a separable quadratic extension (an algebra which is either isomorphic to $K \oplus K$ or isomorphic to a separable quadratic extension field of K is also called an *étale quadratic extension* of K). In both cases, $Z(C_0(q_1))$ does not contain an element z such that $z \notin K$ but $z^2 \in K$.

Let R be the (non-trivial) radical of (V, q_2). Then V has a decomposition $V = R \oplus W$, and $(W, q_2|_W)$ is a regular quadratic form. To avoid confusion, we will denote the image of an element $v \in V$ under the natural embedding of V in $C(V, q_2)$ as $\tilde{v}$. For all $r \in R = \mathrm{Rad}(V, q_2)$ and all $v \in V$, we have that $q_2(r + v) = q_2(r) + q_2(v)$, and hence $(\tilde{r} + \tilde{v})^2 = \tilde{r}^2 + \tilde{v}^2$. It follows that $\tilde{r}\tilde{v} = \tilde{v}\tilde{r}$, so $\tilde{R} \subseteq Z(C(V, q_2))$. Since a regular quadratic form in even characteristic is always even dimensional, $\dim_K W$ is even, and hence $\dim_K R \geq 2$. So we can choose two linearly independent elements $u, v \in R$. It follows from Theorem 1.8.3 that $\tilde{u}$ and $\tilde{v}$ are linearly independent in $C(V, q_2)$ as well. Hence $\dim_K \tilde{R} \geq 2$, and since $\tilde{R} \subseteq Z(C(V, q_2))$, it follows that $\dim_K Z(C(q_2)) \geq 2$, whereas $\dim_K Z(C(q_1)) = 1$. So $Z(C(q_1))$ and $Z(C(q_2))$ cannot be isomorphic.

On the other hand, it also follows from the fact that $\tilde{u}$ and $\tilde{v}$ are linearly independent in $C(V, q_2)$ that $\tilde{u}\tilde{v} \notin K$. But $\tilde{u}$ and $\tilde{v}$ commute, so $(\tilde{u}\tilde{v})^2 = \tilde{u}^2\tilde{v}^2 = q(u)q(v) \cdot 1 \in K$. Thus $Z(C_0(q_2))$ contains an element $z = \tilde{u}\tilde{v}$ such that $z \notin K$ but $z^2 \in K$. We conclude that $Z(C_0(q_1))$ and $Z(C_0(q_2))$ are not isomorphic either, and we are done. □

We can now restate the main theorem.

Theorem 4.3.4.

(i) *Let q be a 6-dimensional anisotropic quadratic form over K. Then the following are equivalent.*

(a) *q is of type E_6.*
(b) *$C_0(q) \cong \mathrm{Mat}_4(E)$ for some quadratic extension E/K.*
(c) *q is regular, $d(q)$ is non-trivial and $c(q) = [Q]$ for some quaternion algebra Q over K which, if it is a division algebra, contains the discriminant extension of q.*

(ii) *Let q be an 8-dimensional anisotropic quadratic form over K. Then the following are equivalent.*

(a) *q is of type E_7.*
(b) *$C_0(q) \cong \mathrm{Mat}_4(D) \oplus \mathrm{Mat}_4(D)$ for some quaternion division algebra D over K.*
(c) *q is regular, $d(q)$ is trivial and $c(q) = [D]$ for some quaternion division algebra D over K.*

(iii) *Let q be a 12-dimensional anisotropic quadratic form over K. Then the following are equivalent.*

(a) *q is of type E_8.*
(b) *$C_0(q) \cong \mathrm{Mat}_{32}(K) \oplus \mathrm{Mat}_{32}(K)$.*
(c) *q is regular, $d(q)$ is trivial and $c(q) = 0$.*

Proof. In each case, we have proved in Theorem 4.1.6 that (a) implies (c) and in Theorem 4.2.12 that (c) implies (a). It remains to show that (b) and (c) are equivalent.

We start by proving that (b) implies (c). First of all, observe that, in all cases, it follows from Lemma 4.3.3(i) that q is a regular quadratic form. In each case, we will reconstruct $d(q)$ and $c(q)$ from $C_0(q)$, using Theorem 4.3.1. By this theorem, we have $C(q) \cong \mathrm{Mat}_k(A)$, where A is a central division algebra over K.

(i) Since $C_0(q)$ is simple, we are in case (ii) of Theorem 4.3.1. In particular, $d(q)$ is non-trivial, and we can consider the discriminant extension $\tilde{E}/K$.

If we are in case (a), then $\tilde{E}$ is contained in A, and $C_0(q) \cong \mathrm{Mat}_k(D_0)$ for some central division algebra $D_0 \subseteq A$ over $\tilde{E}$. Since $C_0(q) \cong \mathrm{Mat}_4(E)$, this implies that $k = 4$, so $C(q) \cong \mathrm{Mat}_4(A)$. By Theorem 1.8.3, $\dim_K A = 4$, so A is a quaternion division algebra. We conclude that $c(q) = [A]$, where A is a quaternion division algebra over K which contains the discriminant extension of q.

If we are in case (b), then $C_0(q) \cong \mathrm{Mat}_{k/2}(A \otimes_K \tilde{E})$, and $A \otimes_K \tilde{E}$ is a division algebra. Since $C_0(q) \cong \mathrm{Mat}_4(E)$, this implies that $k = 8$, so $C(q) \cong \mathrm{Mat}_8(A)$, and Theorem 1.8.3 yields immediately that $A \cong K$. We conclude that $c(q) = 0$ in this case.

(ii) Since $C_0(q)$ is not simple, we are in case (i) of Theorem 4.3.1. In particular, $d(q)$ is trivial, and $A \cong D$, so $C(q) \cong \mathrm{Mat}_8(D)$ and hence $c(q) = [D]$.

(iii) Since $C_0(q)$ is not simple, we are in case (i) of Theorem 4.3.1. In particular, $d(q)$ is trivial, and $A \cong K$, so $C(q) \cong \mathrm{Mat}_{64}(K)$ and hence $c(q) = 0$.

We finish by proving that (c) implies (b).

(i) Since $d(q)$ is non-trivial, we can consider the discriminant extension E/K of q. By Theorem 1.8.3, $C(q) \cong \mathrm{Mat}_4(Q)$. If Q is not a division algebra, i.e. if $Q \cong \mathrm{Mat}_2(K)$, then $C(q) \cong \mathrm{Mat}_8(K)$, and it follows from case (ii.b) of Theorem 4.3.1 that $C_0(q) \cong \mathrm{Mat}_4(E)$. If Q is a division algebra which contains E, then it follows from case (ii.a) of Theorem 4.3.1 that $C_0(q) \cong \mathrm{Mat}_4(D_0)$, for some central division algebra $D_0 \subseteq D$ over E. By counting dimensions using Theorem 1.8.5, we see that $\dim_E D_0 = 1$, hence $D_0 \cong E$.

(ii) By Theorem 1.8.3, $C(q) \cong \mathrm{Mat}_8(D)$. It follows from case (i) of Theorem 4.3.1 that $C_0(q) \cong \mathrm{Mat}_4(D) \oplus \mathrm{Mat}_4(D)$.

(iii) By Theorem 1.8.3, $C(q) \cong \mathrm{Mat}_{64}(K)$. It follows from case (i) of Theorem 4.3.1 that $C_0(q) \cong \mathrm{Mat}_{32}(K) \oplus \mathrm{Mat}_{32}(K)$.

□

Remark 4.3.5. In a completely similar way as in Theorem 4.3.4(ii), one can prove that the following are equivalent for an 8-dimensional anisotropic quadratic form q over K:

(a) q is similar to the norm form of an octonion division algebra.

(b) $C_0(q) \cong \mathrm{Mat}_8(K) \oplus \mathrm{Mat}_8(K)$.

(c) q is regular, $d(q)$ is trivial and $c(q) = 0$.

One should be aware of the following equivalences:

q is similar to the norm form of an octonion division algebra.

$\Longleftrightarrow$ q is anisotropic and $q \simeq \alpha((N - \beta N) - \gamma(N - \beta N))$ for some separable quadratic extension E/K with norm N and some $\alpha, \beta, \gamma \in K^*$; see, for example, [52, (9.8) and (9.3)].

$\Longleftrightarrow$ q is anisotropic and $q \simeq s_1 N \perp s_2 N \perp s_3 N \perp s_4 N$ for some separable quadratic extension E/K with norm N and some $s_1, s_2, s_3, s_4 \in K^*$ with $s_1 s_2 s_3 s_4 \in N(E)$.

Note that this is also equivalent to the statement that "q is similar to an anisotropic 3-fold Pfister form" (in all characteristics). The result 4.3.5 was known before; see, for example, [41, 2.14.4] for a proof in odd characteristic.

A About the LaTeX Code

Just in case you are curious about how this thesis was typeset, I include the main LaTeX file (`phd.tex`) and the LaTeX style file (`tomphd.sty`) which I have made for this purpose.

A.1 phd.tex

In fact, there is nothing special about this main LaTeX file. The preamble is kept as short as possible: it imports only one package (`tomphd.sty`), and it defines a couple of new commands and operators which are used throughout this thesis. (We omitted most of the operators in this listing.) The file `counters.tex` is explained in section A.3. Then each of the chapters is included. There is only one curiosity, namely that the inclusion of the glossary file `phd.gdx` which is automatically generated by the `\makeglossary` command in combination with the `bash` script below (A.4), has some trouble with the '`@`' character. Therefore, the commands `\makeatletter` and `\makeatother` had to be included.

```
%
% This is phd.tex, the main LaTeX file for this thesis.
% By TDM
%
\mag=850
\documentclass[a4paper, 11pt, twoside, openright]{book}
\raggedbottom
\usepackage{tomphd}
\addtolength{\oddsidemargin}{27.6mm}

\makeindex
\makeglossary

\newcommand{\inc}{\mathbf{I}}
\newcommand{\cP}{\mathcal{P}}
\newcommand{\cL}{\mathcal{L}}
```

```
\newcommand{\bJ}{\textbf{J}}
\newcommand{\bH}{\textbf{H}}
\newcommand{\dual}{\mathrm{D}}
\newcommand{\into}{\hookrightarrow}

\DeclareMathOperator{\Aut}{Aut}
[...]
\DeclareMathOperator{\Rad}{Rad}

\newcommand{\pfrac}[2]{\genfrac{(}{)}{}{}{{#1}}{{#2}}}
\newcommand{\bpfrac}[2]{\genfrac{\lbrack}{)}{}{}{{#1}}{{#2}}}

\include{counters} % Create some new some counters for the axiom systems

\begin{document}

\selectlanguage{english}
\hyphenation{Mou-fang}

\frontmatter

\include{Frontpage}
\pagestyle{empty}\cleardoublepage\pagestyle{fancy}
\include{Preface}
\pagestyle{empty}\cleardoublepage\pagestyle{fancy}
\include{Introduction}
\pagestyle{empty}\cleardoublepage\pagestyle{fancy}

\addcontentsline{toc}{chapterlike}{Contents}
{\footnotesize\tableofcontents}

\mainmatter

\include{Chapter1} % --> Preliminaries
\pagestyle{empty}\cleardoublepage\pagestyle{fancy}
\include{Chapter2} % --> Quadrangular Systems
\pagestyle{empty}\cleardoublepage\pagestyle{fancy}
\include{Chapter3} % --> Automorphisms of F4-quadrangles
\pagestyle{empty}\cleardoublepage\pagestyle{fancy}
\include{Chapter4} % --> Quadratic forms of type E6, E7 and E8
\pagestyle{empty}\cleardoublepage\pagestyle{fancy}
\appendix
\include{AppendixA} % --> About the LaTeX code
\pagestyle{empty}\cleardoublepage\pagestyle{fancy}
\include{AppendixB} % --> Nederlandstalige samenvatting
\pagestyle{empty}\cleardoublepage\pagestyle{fancy}

\addcontentsline{toc}{chapterlike}{Bibliography}
\include{Bibliography}

\addcontentsline{toc}{chapterlike}{Index}
\footnotesize
\printindex

\renewcommand{\indexname}{{List of Notations}}
\addcontentsline{toc}{chapterlike}{List of Notations}
\makeatletter
\input{phd.gdx}
\makeatother

\end{document}
```

A.2 tomphd.sty

The style file `tomphd.sty` is somewhat more complicated. First of all, it includes a bunch of packages. The commands `\defin` and `\notat` write out the main argument in the text and add it to the Index or to the List of Notations, respectively. If a second (optional) argument is given, then this argument is added to the list instead of the main argument which is shown in the text. Then a couple of standard environments are defined. The `\bysame` command is used in the Bibliography.

The "`Very Fancy Headings`" section which follows, makes use of the `fancyhdr` and the `pstricks` and `pst-node` packages to produce the headings on top of each page. The `\gradline` command, which makes use of the `pst-grad` package, produces the fancy line on each chapter page; the `\gradindexline` command does the same for the index at the end. The following paragraph redefines the `\itemize` command to decrease the amount of space between subsequent items (which is quite large by default). Then the `Table of Contents` is attacked. It adds a horizontal line before and after each chapter entry – I had quite some trouble to make the distance independent of the "depth" of the letters (in the sense that a 'g' is deep and an 'a' is not); this was solved by using a so-called strut, i.e. a line of width zero. Also note the one-liner to adjust the alignment for the page numbers above 100.

Finally, the "`Nicer Chapter Headings`" part is responsible for the beginning of each Chapter. Then all the section commands are redefined; the only change compared to the original definitions are the `\boldmath` commands which I have added to make sure that the titles look nicer if there is some math within the title.

```
%
% This is tomphd.sty, the LaTeX style file for this thesis.
% By TDM
%
\ifx\pdfoutput\undefined
    \usepackage[dvips]{graphicx}
\else
    \usepackage[pdftex]{graphicx}
\fi
\usepackage[english,dutch]{babel}
\usepackage{ifthen}
\usepackage{amsmath}
\usepackage{amssymb}
\usepackage{amsthm}
\usepackage{fancyhdr}
\usepackage{color}
\usepackage{makeidx}
\usepackage{hyphenat}
\usepackage{mathpple}           % Math Palatino
\usepackage{charter}            % Text Charter
```

```
\renewcommand{\sfdefault}{pag}  % Use a nicer font for the headers
\usepackage{pstricks}
\usepackage{pst-grad}
\usepackage{pst-node}

\newcommand{\defin}[2][]{%
        \ifthenelse{\equal{#1}{}}
                {{\em{#2}\index{#2}}}%
                {{\em{#2}\index{#1}}}%
}
\newcommand{\notat}[2][]{%
        \ifthenelse{\equal{#1}{}}
                {{#2}\glossary{$#2$}}%
                {{#2}\glossary{$#1$}}%
}
\newcommand{\glossaryentry}[2]{\item {#1}, {#2}}

\newtheorem{theorem}{Theorem}[section]
\newtheorem{maintheorem}[theorem]{Main Theorem}
\newtheorem{lemma}[theorem]{Lemma}
\newtheorem{stelling}[theorem]{Stelling}

\theoremstyle{definition}
\newtheorem{definition}[theorem]{Definition}
\newtheorem{definitie}[theorem]{Definitie}

\theoremstyle{remark}
\newtheorem{remark}[theorem]{Remark}
\newtheorem{impremark}[theorem]{Important Remark}
\newtheorem{opmerking}[theorem]{Opmerking}

\def\bysame{\leavevmode\hbox to3em{\hrulefill}\thinspace}


%
% Very Fancy Headings ;-)
% By TDM
%
\pagestyle{fancy}
\setlength{\headheight}{23.03pt}
\renewcommand{\chaptermark}[1]{\markboth{\sf\thechapter.\ #1}{}}
\renewcommand{\sectionmark}[1]{\markright{\sf\thesection.\ #1}{}}
\ifx\pdfoutput\undefined
        \newcommand{\whitebox}[1]{\psframebox[linewidth=.3pt]{#1}}
        \newcommand{\blackbox}[1]{\psframebox[fillstyle=solid, fillcolor=black, %
                                  linewidth=.3pt]{{\textcolor{white}{#1}}}}
\else
        \newcommand{\whitebox}[1]{\fbox{#1}}
        \newcommand{\blackbox}[1]{\fbox{#1}}
\fi
\fancyhf{}
\fancyhead[el]{\Rnode{Pl}{\blackbox{\sf\thepage\rule[-0.55ex]{0pt}{2.25ex}}}}
\fancyhead[or]{\psset{linewidth=.3pt}\Rnode{Pr}{\blackbox{\sf\thepage%
        \rule[-0.55ex]{0pt}{2.25ex}}}\ncline{Tr}{Pr}}

% The chapter on the even (=left) pages; the section on the odd (=right) pages.
\fancyhead[ec]{\fancyplain{}{\psset{linewidth=.3pt}\Rnode{Tl}{\whitebox{\ %
        \nouppercase{\sf\leftmark}\ \rule[-0.6ex]{0pt}{2.5ex}}}}\ncline{Pl}{Tl}}
\fancyhead[oc]{\fancyplain{}{\Rnode{Tr}{\whitebox{\rule[-0.6ex]{0pt}{2.5ex}\ %
        \nouppercase{\sf\rightmark}\ }}}}
\renewcommand{\headrulewidth}{0pt}        % Get rid of the line.
```

```
\fancypagestyle{plain}{%                  % This is for the chapter pages.
    \fancyhf{}                            % No headers or footers ...
    \renewcommand{\headrulewidth}{0pt}    % ... and get rid of the line.
}


%
% A cool "fading out" line on each chapter page
% By TDM
%
\newcommand{\gradline}{%
\psset{linecolor=white, linewidth=0pt, fillstyle=gradient, %
gradbegin=white, gradend=darkgray, gradmidpoint=0}
\pspicture(0,0)(14,.1)
\psframe[gradangle=90](0,0)(14,.05)
\endpspicture
}
\newcommand{\gradindexline}[1]{%
\psset{linecolor=white, linewidth=0pt, fillstyle=gradient, %
gradbegin=white, gradend=darkgray, gradmidpoint=0}
\begin{pspicture}(0,-.3)(6,.4)
\rput[l](0,0){#1}
\psframe[gradangle=90](.8,-.1)(6,-.05)
\end{pspicture}
}


\makeatletter
% This command usually means that something freaky is going to happen... ;-)


%
% Smaller separation between items
% Inspired by "tweaklist.sty"
%
\renewcommand{\itemize}{%
  \ifnum \@itemdepth >\thr@@\@toodeep\else
    \advance\@itemdepth\@ne
    \edef\@itemitem{labelitem\romannumeral\the\@itemdepth}%
    \expandafter
    \list
      \csname\@itemitem\endcsname
      {\def\makelabel##1{\hss\llap{##1}}%
        \topsep=.8ex\itemsep=-.2ex}%
  \fi%
}


%
% Lines under and above Chapters in the TOC
% By TDM
%
\newcommand*\l@chapterlike[2]{%
  \ifnum \c@tocdepth >\m@ne
    \addpenalty{-\@highpenalty}%
    \vskip 1.6em \@plus\p@
    \setlength\@tempdima{1.5em}%
    \begingroup
      \parindent \z@ \rightskip \@pnumwidth
      \parfillskip -\@pnumwidth
      \leavevmode \boldmath\bfseries
```

```
      \advance\leftskip\@tempdima
      \hskip -\leftskip
      \ #1\nobreak\hfil \nobreak\hb@xt@\@pnumwidth{\hss #2}\par
      \penalty\@highpenalty
    \endgroup
  \fi%
}


\renewcommand*\l@chapter[2]{%
  \ifnum \c@tocdepth >\m@ne
    \addpenalty{-\@highpenalty}%
    \vskip 1.4em \@plus\p@
    \setlength\@tempdima{1.5em}%
    \hrule
    \vskip 0.5em \@plus\p@
    \begingroup
      \parindent \z@ \rightskip \@pnumwidth
      \parfillskip -\@pnumwidth
      \leavevmode \boldmath\bfseries
      \advance\leftskip\@tempdima
      \hskip -\leftskip
      \ #1\nobreak\hfil \nobreak\hb@xt@\@pnumwidth%
            {\hss \rule[-0.5ex]{0pt}{2ex} #2}\par         % Note the strut!
      \penalty\@highpenalty
    \endgroup
    \vskip 0.4em \@plus\p@
    \hrule
    \vskip 0.5em \@plus\p@
  \fi}
\renewcommand\@pnumwidth{1.75em}  % Better right alignment in the TOC (>100 pages).


%
% Nicer Chapter headings
% By TDM
%
\definecolor{gray}{rgb}{.5, .5, .5}
\font\veryhuge=pbkd8r at 100pt

\def\@makechapterhead#1{%
  \vspace*{10\p@}%
  {\parindent \z@ \raggedright \normalfont
    \ifnum \c@secnumdepth >\m@ne
      \if@mainmatter
        \begin{minipage}[b]{3cm}
           {\veryhuge\textcolor{gray}{\thechapter}}
        \end{minipage}
      \fi
    \fi
    \begin{minipage}[b]{9cm}
        \Huge\bfseries \boldmath \nohyphens{#1}\par\nobreak
        \vspace{0\p@}
    \end{minipage}
    \vskip 30\p@
    \gradline
    \vskip 50\p@
  }%
}
```

```
%
% Make sure that mathematics are in bold in all title headings
% By TDM
%
\renewcommand\section{\@startsection {section}{1}{\z@}%
                             {-3.5ex \@plus -1ex \@minus -.2ex}%
                             {2.3ex \@plus.2ex}%
                             {\boldmath\normalfont\Large\bfseries}}
\renewcommand\subsection{\@startsection{subsection}{2}{\z@}%
                             {-3.25ex\@plus -1ex \@minus -.2ex}%
                             {1.5ex \@plus .2ex}%
                             {\boldmath\normalfont\large\bfseries}}

\renewcommand\subsubsection{\@startsection{subsubsection}{3}{\z@}%
                             {-3.25ex\@plus -1ex \@minus -.2ex}%
                             {1.5ex \@plus .2ex}%
                             {\boldmath\normalfont\normalsize\bfseries}}
\renewcommand\paragraph{\@startsection{paragraph}{4}{\z@}%
                             {3.25ex \@plus1ex \@minus.2ex}%
                             {-1em}%
                             {\boldmath\normalfont\normalsize\bfseries}}
\renewcommand\subparagraph{\@startsection{subparagraph}{5}{\parindent}%
                             {3.25ex \@plus1ex \@minus .2ex}%
                             {-1em}%
                             {\boldmath\normalfont\normalsize\bfseries}}

\makeatother
```

A.3 counters.tex

Now for the counters. The following lump of code creates a **Q**-counter, and defines the commands `\Qitem` (which takes a label as an optional argument) and `\Qref` (which refers to such a label). For example, the code

```
\begin{itemize}
    \Qitem
        This is a Q-item.
    \Qitem[second]
        This is a second Q-item.
\end{itemize}
```

then produces

($\mathbf{Q}_{27}$) This is a Q-item.

($\mathbf{Q}_{28}$) This is a second Q-item.

and the command `\Qref{second}` now produces ($\mathbf{Q}_{28}$). The `counters.tex` file simply contains this code together with completely similar code for the letters **A**, **R**, **D**, **F** and **C** :

```
\newcounter{Q}
\newcommand{\Qitem}[1][]{%
    \ifthenelse{\equal{#1}{}}
        {\item[(${\bf Q}_{\refstepcounter{Q}\theQ}$)]}
        {\item[(${\bf Q}_{\refstepcounter{Q}\label{qitem:#1}\theQ}$)]}
}
\newcommand{\Qref}[1]{(${\bf Q}_{\ref{qitem:#1}}$)}
```

A.4 Producing the Index and the List of Notations

In order to produce the Index table and the List of Notations, the following `bash` script was used.

```
#!/bin/bash
latex phd.tex
makeindex -s phd.ist phd
echo "\begin{theindex}" > phd.gdx
echo >> phd.gdx
cat phd.glo >> phd.gdx
echo >> phd.gdx
echo "\end{theindex}" >> phd.gdx
latex phd.tex
```

The file `phd.ist` contains the information to generate the big capital letters and the lines in the Index, and reads as follows.

```
headings_flag 1
heading_prefix "\n{\\LARGE\\sffamily\\bfseries\\gradindexline{"
heading_suffix "}}\n"
```

And that's it!

B Nederlandstalige Samenvatting

B.1 Inleiding

Het begrip "veralgemeende veelhoek" vinden we voor het eerst terug in een erg korte appendix van een lang en moeilijk – en intussen beroemd geworden – artikel, Tits 1959 [42]. In dit artikel heeft Jacques Tits de enkelvoudige groepen van type $^3D_4$ ontdekt door de trialiteiten van een D_4-meetkunde te klassificeren die minstens één absoluut punt hebben. De methode die hij hiervoor gebruikte, was hoofdzakelijk meetkundig van inslag, en het is dan ook niet verwonderlijk dat de corresponderende meetkundes hier een belangrijke rol in speelden. Dit was het ontstaan van de veralgemeende zeshoeken. Natuurlijk is 1959 slechts een officiële 'geboortedatum', want veralgemeende vierhoeken werden al langer bestudeerd, zij het dan als bijvoorbeeld kwadrieken van Witt index 2 of als rechtensystemen corresponderend met symplectische polariteiten in een driedimensionale projectieve ruimte over een veld. Ook veralgemeende driehoeken waren al lang bekend, want dit zijn precies de projectieve vlakken. Maar het idee om echt meetkundes zoals veralgemeende veelhoeken op zichzelf te gaan bestuderen, komt van Tits.

Tegelijk vormen de veralgemeende veelhoeken ook een zeer specifieke klasse van gebouwen, met name de sferische gebouwen van rang 2. De sferische gebouwen van rang tenminste 3 zijn reeds in 1974 door Tits volledig geklassificeerd in [44]. Daarin wordt bewezen dat elk irreduciebel sferisch gebouw van rang ≥ 3 geassocieerd is met een algebraïsche enkelvoudige groep, een klassieke groep of een gemixte groep van type F_4.

Daarentegen kan men niet verwachten dat de sferische gebouwen van rang 2 (dus de veralgemeende veelhoeken) volledig te klassificeren zijn. Immers, er bestaan vrije constructies van veralgemeende n-hoeken (zie bijvoorbeeld Tits [47]), wat meestal een klassificatie onmogelijk maakt; ook merken we op dat de projectieve vlakken, die een deelklasse vormen van de veralgemeende veelhoeken, zelfs in het eindige geval nog niet geklassificeerd zijn. Bijgevolg is men op zoek gegaan naar een bijkomende voorwaarde onder dewelke de klassificatie toch mogelijk zou zijn. De zogenaamde Moufang-voorwaarde bleek in een aantal opzichten de geschikte keuze.

Ten eerste is het zo dat de veralgemeende veelhoeken die ook sferische gebouwen zijn geassocieerd met een klassieke of met een enkelvoudige algebraïsche groep, allen voldoen aan de Moufang voorwaarde.

Ten tweede vormen de Moufang veelhoeken een heel natuurlijke uitbreiding van de Moufang vlakken, die al een hele tijd tevoren werden bestudeerd door Ruth Moufang (zie [33]); de benaming "Moufang vlak" is afkomstig van G. Pickert [38].

Ten derde is de Moufang voorwaarde ook een lichtjes verzwakte vorm van de zogenaamde Steinberg relaties (zie [51]).

Het vermoeden dat de veralgemeende veelhoeken die aan de Moufang voorwaarde voldoen (die we kortweg Moufang veelhoeken noemen) kunnen geklassificeerd worden, werd voor het eerst geopperd door Tits in [44], en kwam al heel wat systematischer naar voor in [45]. Tits had echter één bepaalde klasse over het hoofd gezien, met name de Moufang vierhoeken van type F_4. De klassificatie van de Moufang achthoeken, en het feit dat Moufang n-hoeken enkel bestaan voor $n \in \{3, 4, 6, 8\}$, werd bewezen in [46], [48], [49] en [54]. Zoals Tits reeds van in het begin had vastgesteld, volgt de klassificatie van de *eindige* Moufang veelhoeken gemakkelijk uit het werk van P. Fong en G. Seitz [18, 19] ten gevolge van hun klassificatie van de eindige "gespleten BN-paren" van rang 2.

In 1996 hebben Jacques Tits en Richard Weiss samen de volledige klassificatie van de Moufang veelhoeken aangevat, op een manier die volledig elementair is, en die geen enkel verband vertoont met de methode van Fong en Seitz. Deze klassificatie is intussen voltooid, en is verschenen in de vorm van een boek "Moufang Polygons" [52]. Verrassend genoeg heeft R. Weiss tijdens deze klassificatie een nieuwe klasse van Moufang vierhoeken ontdekt, die dan een korte tijd later door Bernhard Mühlherr en Hendrik Van Maldeghem zijn herkend in het gebouw van type F_4 (zie [34]).

Voor $n \in \{3, 6, 8\}$ is het bewijs mooi onderverdeeld in twee delen, met name (A) het aantonen dat een Moufang n-hoek kan beschreven worden door een welbepaalde algebraïsche structuur, en (B) het klassificeren van deze algebraïsche structuur. Meer bepaald heeft R. Moufang in 1933 reeds bewezen, zij het onder een ietwat andere gedaante[1], dat alle Moufang driehoeken geparametrizeerd kunnen worden door een alternatieve delingsring, een begrip dat door M. Zorn[2] werd ingevoerd in [55]. Deze alternatieve delingsringen zijn in 1951 geklassificeerd door R. Bruck en E. Kleinfeld; zie [10]. De Moufang zeshoeken kunnen alle beschreven worden door een unitale kwadratische Jordan delingsalgebra van graad 3, ook gekend onder de naam anisotrope kubische normstructuur; zie [45]. Deze structuren zijn in hun volledige algemeenheid geklassificeerd in 1986 door H. Petersson en M. Racine [36, 37]; hun bewijs is verdergebouwd op vroeger werk van A. Albert [2], F.D. Jacobson en N. Jacobson [23], N. Jacobson [24, 25] en K. McCrimmon [31, 32]. De Moufang achthoeken tenslotte kunnen beschreven worden door een zogenaamd octagonaal systeem, zoals is aangetoond door J. Tits in 1983 (zie [49]); aangezien deze systemen een heel eenvoudige beschrijving hebben, is er in dit geval geen nood aan een (B)-gedeelte.

De klassificatie van de Moufang vierhoeken ($n = 4$) in [52] is niet op deze manier georganiseerd, bij gebrek aan een algemene algebraïsche structuur om deze te beschrijven. In plaats daarvan worden deze vierhoeken beschreven door zes verschillende parametersystemen, en zelfs dan is het onderscheid tussen de delen (A) en (B) niet aanwezig in de twee gevallen die tot de exceptionele vierhoeken leiden.

Het tweede hoofdstuk van deze thesis, dat tevens ook het beredeel van dit werk uitmaakt, heeft dan ook als doel om een dergelijke overkoepelende algebraïsche structuur te presenteren. Deze “quadrangulaire systemen”, zoals we deze noemen, scheppen een nieuw licht op deze vierhoeken, en brengen structuur in deze vierhoeken naar boven die zonder deze systemen moeilijk te zien is. Zo hebben we deze structuren bijvoorbeeld met succes gebruikt om een belangrijke vraag op te lossen over de automorfismengroep van de exceptionele Moufang vierhoeken van type F_4, die in [52] werd opengelaten; zie hoofdstuk 3. Bovendien is het mogelijk om deze structuren volledig te klassificeren zonder daarbij gebruik te maken van de Moufang vierhoeken waaruit deze ontstaan zijn. Op die manier hebben we tegelijk een nieuw bewijs geleverd voor de klassificatie van de Moufang vierhoeken dat wel degelijk bestaat uit de gedeelten (A) en (B).

In het laatste hoofdstuk bekijken we de exceptionele Moufang vierhoeken van type E_6, E_7 en E_8 vanuit een ander standpunt. Het is reeds duidelijk vanuit de constructie dat de even Clifford algebra van de kwadratische vorm die deze vierhoeken bepaalt, een belangrijke rol speelt in het “be-

[1] Zie de lange voetnoot op pagina 176 in [52].

[2] Jawel, die van dat Lemma.

grijpen" van de structuur van deze vierhoeken. We trekken dit nog verder door, door aan te tonen dat dergelijke kwadratische vormen *volledig* gekarakteriseerd worden door de structuur van deze even Clifford algebra alleen.

B.2 Quadrangulaire Systemen

Definitie B.2.1. Een *veralgemeende n-hoek* Γ is een samenhangend bipartitie-graaf met diameter n en omtrek $2n$. Elk circuit van lengte $2n$ wordt een *appartement* van Γ genoemd. Elk n-pad in Γ wordt een *half appartement* of een *wortel* van Γ genoemd. We kunnen Γ ook als meetkunde bekijken. Een veralgemeende n-hoek Γ is dan een incidentiemeetkunde $(\mathcal{P}, \mathcal{L}, \mathbf{I})$ met de eigenschap dat Γ geen gewone k-hoeken bevat voor alle $k \in \{2, \ldots, n-1\}$, en zodat door elke twee elementen $x, y \in \mathcal{P} \cup \mathcal{L}$ minstens één n-hoek gaat. We noemen een veralgemeende n-hoek *dik* als elke top adjacent is met minstens 3 andere toppen.

Definitie B.2.2. Zij Γ een veralgemeende n-hoek voor een bepaalde $n \geq 3$, en zij $\alpha = (x_0, x_1, \ldots, x_{n-1}, x_n)$ een willekeurige wortel van Γ. De verzameling bestaande uit alle automorfismen van Γ die alle toppen fixeren die adjacent zijn met minstens één element van $\{x_1, \ldots, x_{n-1}\}$, vormt een groep, die we noteren als U_α en die we de *wortelgroep* corresponderende met de wortel α noemen. De groep U_α werkt semi-regulier op de verzameling van alle appartementen door α. Indien deze actie ook transitief (en dus regulier) is, zeggen we dat α een *Moufang wortel* is van Γ. Indien alle wortels in Γ Moufang wortels zijn, dan noemen we Γ een *Moufang veelhoek*, of we zeggen dat Γ aan de *Moufang voorwaarde* voldoet.

Veronderstel vanaf nu dat Γ een dikke Moufang n-hoek is voor een bepaalde $n \geq 3$. Fixeer een willekeurig appartement Σ; we zullen dit labelen met de gehele getallen modulo $2n$, dus $\Sigma = \{0, \ldots, 2n-1\}$, waarbij i en $i+1$ steeds adjacente toppen zijn. Voor elke wortel $\alpha_i := \{i, i+1, \ldots, i+n\}$ in Σ definiëren we nu $U_i := U_{\alpha_i}$. Merk op dat deze wortelgroepen niet triviaal zijn omdat Γ dik is. We definiëren ook nog

$$U_{[i,j]} := \begin{cases} \langle U_i, U_{i+1}, \ldots, U_j \rangle & \text{als } i \leq j < i+n \,; \\ 1 & \text{anders} \,. \end{cases}$$

De volgende verrassende stelling is van fundamenteel belang voor de klassificatie van de Moufang veelhoeken.

Stelling B.2.3. *Een Moufang veelhoek is uniek bepaald door het* $(n+1)$*-tal*

$$(U_{[1,n]}, U_1, U_2, \ldots, U_n)\ .$$

Uiteraard zal niet elke groep $U_{[1,n]}$ die voortgebracht is door deelgroepen $U_1, \ldots, U_n$ aanleiding geven tot een Moufang veelhoek. Indien dit wél het geval is, dan wordt het $(n+1)$-tal $(U_{[1,n]}, U_1, U_2, \ldots, U_n)$ een *wortelgroepenreeks* genoemd. De volgende stelling geeft de nodige en voldoende voorwaarden waaraan zo'n $(n+1)$-tal moet voldoen.

Stelling B.2.4. *Veronderstel dat* $U_{[1,n]}$ *een groep is voortgebracht door niettriviale deelgroepen* $U_1, \ldots, U_n$ *, zodanig dat de volgende axioma's gelden.*

$\mathcal{M}_1$. $[U_i, U_j] \leq U_{[i+1,j-1]}$ *voor* $1 \leq i < j \leq n$.

$\mathcal{M}_2$. *De produktafbeelding van* $U_1 \times \cdots \times U_n$ *naar* $U_{[1,n]}$ *is bijectief.*

$\mathcal{M}_3$. *Er bestaat een deelgroep* $\tilde{U}_0$ *van* $\mathrm{Aut}(U_{[1,n-1]})$ *zodat er voor elke* $a_n \in U_n^*$ *een* $\mu(a_n) \in \tilde{U}_0^* \tilde{a}_n \tilde{U}_0^*$ *bestaat zodat* $U_j^{\mu(a_n)} = U_{n-j}$ *voor* $1 \leq j \leq n-1$, *en zodat er voor* <u>*een*</u> $e_n \in U_n^*$ *bovendien geldt dat* $\tilde{U}_j^{\mu(e_n)} = \tilde{U}_{n-j}$ *voor* $j = 0$ *en* $j = n$.

$\mathcal{M}_4$. *Er bestaat een deelgroep* $\tilde{U}_{n+1}$ *van* $\mathrm{Aut}(U_{[2,n]})$ *zodat er voor elke* $a_1 \in U_1^*$ *een* $\mu(a_1) \in \tilde{U}_{n+1}^* \tilde{a}_1 \tilde{U}_{n+1}^*$ *bestaat zodat* $U_j^{\mu(a_1)} = U_{n+2-j}$ *voor* $2 \leq j \leq n$, *en zodat er voor* <u>*een*</u> $e_1 \in U_1^*$ *bovendien geldt dat* $\tilde{U}_j^{\mu(e_1)} = \tilde{U}_{n+2-j}$ *voor* $j = 1$ *en* $j = n+1$.

Dan bestaat er een Moufang n*-hoek die* $(U_{[1,n]}, U_1, U_2, \ldots, U_n)$ *heeft als wortelgroepenreeks. Bovendien voldoet elke wortelgroepenreeks van een Moufang* n*-hoek aan de voorwaarden* $(\mathcal{M}_1)$ – $(\mathcal{M}_4)$.

Dankzij de voorwaarden $(\mathcal{M}_1)$ en $(\mathcal{M}_2)$ kunnen we nog een stap verder gaan. We kunnen namelijk de groepen $U_1, \ldots, U_n$ beschrijven, en vervolgens de overkoepelende groep $U_{[1,n]}$ beschrijven door de commutatierelaties tussen de groepen $U_1, \ldots, U_n$ onderling weer te geven. Precies door deze twee voorwaarden $(\mathcal{M}_1)$ en $(\mathcal{M}_2)$ zal de groepsstructuur op $U_{[1,n]}$ daardoor uniek gedefinieerd zijn. Het is deze strategie die tijdens de klassificatie wordt toegepast, en die ook wij zullen toepassen voor het beschrijven van de Moufang vierhoeken.

Definitie B.2.5. Beschouw een abelse groep $(V, +)$ en een al dan niet abelse groep $(W, \boxplus)$. Het inverse van een element $w \in W$ zal worden genoteerd als $\boxminus w$, en de uitdrukking $w_1 \boxplus (\boxminus w_2)$ zullen we verkort weergeven als $w_1 \boxminus w_2$. Veronderstel dat er een afbeelding τ_V van $V \times W$ naar

V is, evenals een afbeelding τ_W van $W \times V$ naar W; we zullen beide afbeeldingen noteren met $\cdot$ of gewoon de elementen aan elkaar schrijven, dus $\tau_V(v,w) = vw = v \cdot w$ en $\tau_W(w,v) = wv = w \cdot v$ voor alle $v \in V$ en alle $w \in W$. Beschouw nu een afbeelding F van $V \times V$ naar W en een afbeelding H van $W \times W$ naar V, beide “bi-additief” in de betekenis dat

$$\begin{aligned} F(v_1 + v_2, v) &= F(v_1, v) \boxplus F(v_2, v)\,; \\ F(v, v_1 + v_2) &= F(v, v_1) \boxplus F(v, v_2)\,; \\ H(w_1 \boxplus w_2, w) &= H(w_1, w) + H(w_2, w)\,; \\ H(w, w_1 \boxplus w_2) &= H(w, w_1) + H(w, w_2)\,; \end{aligned}$$

voor alle $v, v_1, v_2 \in V$ en alle $w, w_1, w_2 \in W$. Veronderstel bovendien dat er een vast element $\epsilon \in V^*$ en een vast element $\delta \in W^*$ bestaan, en veronderstel dat er voor elke $v \in V^*$ een element $v^{-1} \in V^*$ bestaat, en voor elke $w \in W^*$ een element $\kappa(w) \in W^*$ bestaat, zodanig dat voor alle $w, w_1, w_2 \in W$ en alle $v, v_1, v_2 \in V$ de volgende axioma's voldaan zijn. We definiëren

$$\begin{aligned} \overline{v} &:= \epsilon F(\epsilon, v) - v \\ \mathrm{Rad}(F) &:= \{v \in V \mid F(v, V) = 0\} \\ \mathrm{Rad}(H) &:= \{w \in W \mid H(w, W) = 0\} \\ \mathrm{Im}(F) &:= F(V, V) \\ \mathrm{Im}(H) &:= H(W, W) \end{aligned}$$

(Q$_1$) $w\epsilon = w$.

(Q$_2$) $v\delta = v$.

(Q$_3$) $(w_1 \boxplus w_2)v = w_1 v \boxplus w_2 v$.

(Q$_4$) $(v_1 + v_2)w = v_1 w + v_2 w$.

(Q$_5$) $w(-\epsilon) \cdot v = w(-v)$.

(Q$_6$) $v \cdot w(-\epsilon) = vw$.

(Q$_7$) $\mathrm{Im}(F) \subseteq \mathrm{Rad}(H)$.

(Q$_8$) $[w_1, w_2 v]_{\boxplus} = F(H(w_2, w_1), v)$.

(Q$_9$) $\delta \in \mathrm{Rad}(H)$.

(Q$_{10}$) Als $\mathrm{Rad}(F) \neq 0$, dan $\epsilon \in \mathrm{Rad}(F)$.

(Q$_{11}$) $w(v_1 + v_2) = wv_1 \boxplus wv_2 \boxplus F(v_2 w, v_1)$.

(Q$_{12}$) $v(w_1 \boxplus w_2) = vw_1 + vw_2 + H(w_2, w_1 v)$.

(Q$_{13}$) $(v^{-1})^{-1} = v$ (als $v \neq 0$).

$(\mathbf{Q}_{14})$ $\kappa(\boxminus\kappa(\boxminus w)) = w(-\epsilon)$ (als $w \neq 0$).

$(\mathbf{Q}_{15})$ $wv \cdot v^{-1} = w$ (als $v \neq 0$).

$(\mathbf{Q}_{16})$ $v^{-1} \cdot wv = -\overline{v(\boxminus w)}$ (als $v \neq 0$).

$(\mathbf{Q}_{17})$ $F(v_1^{-1}, \overline{v_2})v_1 = F(v_1, v_2)$ (als $v_1 \neq 0$).

$(\mathbf{Q}_{18})$ $v\kappa(w) \cdot (\boxminus w) = -v$ (als $w \neq 0$).

$(\mathbf{Q}_{19})$ $w \cdot v\kappa(w) = \kappa(w)v$ (als $w \neq 0$).

$(\mathbf{Q}_{20})$ $H(\kappa(w_1), w_2)w_1 = H(w_1, w_2)$ (als $w_1 \neq 0$).

Dan noemen we het systeem $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ een *quadrangulair systeem*.

Merk op dat we de functies F en H niet vermelden in onze notatie, evenmin als de afbeeldingen $v \mapsto v^{-1}$ en $w \mapsto \kappa(w)$. De reden is dat deze uniek bepaald zijn door V, W, τ_V, τ_W, ϵ en δ.

Opmerking B.2.6. Bij het neerschrijven van deze axioma's hebben we de conventie gebruikt dat de afbeeldingen die genoteerd zijn door het aan elkaar schrijven voorrang hebben op diegene die met een '$\cdot$' zijn weergegeven. Merk echter op dat er sowieso geen gevaar voor verwarring is, aangezien we geen vermenigvuldiging hebben gedefinieerd op V of op W; vandaar dat we bijvoorbeeld toch dikwijls wvv^{-1} zullen schrijven in plaats van $wv \cdot v^{-1}$.

De volgende twee identiteiten zijn geldig voor elk quadrangulair systeem $(V, W, \tau_V, \tau_W, \epsilon, \delta)$, voor alle $v_1, v_2 \in V$ en alle $w_1, w_2 \in W$.

$(\mathbf{Q}_{21})$ $F(v_1, v_2) = F(v_2, v_1)$.

$(\mathbf{Q}_{22})$ $H(w_1, w_2) = -\overline{H(w_2, w_1)}$.

In zekere zin zeggen deze twee identiteiten dat F een symmetrische vorm is en dat H een scheef-hermitische vorm is. Merk echter op dat V en W in het algemeen *geen* vectorruimten zijn.

Bovendien zijn de volgende vier identiteiten steeds vervuld in een quadrangulair systeem $(V, W, \tau_V, \tau_W, \epsilon, \delta)$, voor alle $v, c \in V$ en alle $w, z \in W$. We veralgemenen eerst het begrip "reflectie" dat we kennen in een kwadratische ruimte.

$$\pi_v(c) := c - vF(v^{-1}, \overline{c}) \qquad \text{(als } v \neq 0\text{)}$$
$$\Pi_w(z) := z \boxplus w(-H(\kappa(w), z)) \qquad \text{(als } w \neq 0\text{)} .$$

Dan zal

$(\mathbf{Q}_{23})$ $v \cdot \Pi_w(z) = -\overline{\overline{\overline{\overline{v}(\boxminus w)}z}\kappa(w)}$ (als $w \neq 0$).

$(\mathbf{Q}_{24})$ $w \cdot \overline{\pi_v(\epsilon)^{-1}} \cdot \overline{\pi_v(c)} = wvcv^{-1}$ (als $v \neq 0$).

($\mathbf{Q}_{25}$) $\pi_v(\overline{c \cdot \delta v})w = \pi_v(\overline{c \cdot wv})$ (als $v \neq 0$).

($\mathbf{Q}_{26}$) $\Pi_{\boxminus z}(w \cdot \epsilon z)v = \Pi_{\boxminus z}(w \cdot vz)$ (als $w \neq 0$).

Veronderstel nu dat $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ een willekeurig quadrangulair systeem is. Zij U_1 en U_3 twee groepen isomorf met W, en zij U_2 en U_4 twee groepen isomorf met V. Noteer de corresponderende isomorfismen als

$$\begin{aligned} x_1 &: W \to U_1 : w \mapsto x_1(w) \; ; \\ x_2 &: \; V \to U_2 : v \mapsto x_2(v) \; ; \\ x_3 &: W \to U_3 : w \mapsto x_3(w) \; ; \\ x_4 &: \; V \to U_4 : v \mapsto x_4(v) \; ; \end{aligned}$$

we zeggen ook dat U_1 en U_3 *geparametrizeerd* zijn door W en dat U_2 en U_4 *geparametrizeerd* zijn door V. We definiëren nu impliciet de groep $U_{[1,4]}$ door middel van de volgende commutatierelaties:

$$\begin{aligned} [x_1(w_1), x_3(w_2)^{-1}] &= x_2(H(w_1, w_2)) \; , \\ [x_2(v_1), x_4(v_2)^{-1}] &= x_3(F(v_1, v_2)) \; , \\ [x_1(w), x_4(v)^{-1}] &= x_2(vw)x_3(wv) \; , \\ [U_i, U_{i+1}] &= 1 \quad \forall i \in \{1, 2, 3\} \; , \end{aligned} \tag{B.1}$$

voor alle $w, w_1, w_2 \in W$ en alle $v, v_1, v_2 \in V$.

De volgende stelling, die we bewijzen in de eerste vier secties van hoofdstuk 2, leert ons dat de studie van de Moufang vierhoeken equivalent is met de studie van de quadrangulaire systemen.

Stelling B.2.7. (i) *Zij $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ een willekeurig quadrangulair systeem, en definieer de groepen $U_1, \ldots, U_4$ en $U_{[1,4]}$ zoals hierboven. Dan bestaat er een Moufang vierhoek Γ die $(U_{[1,4]}, U_1, U_2, U_3, U_4)$ heeft als wortelgroepenreeks.*

(ii) *Zij Γ een willekeurige Moufang vierhoek. Dan bestaat er een quadrangulair systeem $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ zodanig dat de groepen $U_1, \ldots, U_4$ en $U_{[1,4]}$ zoals hierboven gedefinieerd een wortelgroepenreeks vormen van Γ.*

Zonder in detail te gaan geven we een overzicht van de zes verschillende klassen van quadrangulaire systemen; zie sectie 2.6 voor een gedetailleerde beschrijving.

- De quadrangulaire systemen van **kwadratisch type** worden geconstrueerd uit een willekeurige anisotrope kwadratische ruimte (K, V, q); het corresponderend quadrangulair systeem wordt in dit geval genoteerd als $\Omega_Q(K, V, q)$.

- De quadrangulaire systemen van **involutorisch type** worden geconstrueerd vanuit een willekeurig involutiesysteem (K, K_0, σ); het corresponderend quadrangulair systeem wordt in dit geval genoteerd als $\Omega_I(K, K_0, \sigma)$.
- De quadrangulaire systemen van **indifferentieel type** worden geconstrueerd vanuit een willekeurig indifferentiesysteem (K, K_0, L_0); het corresponderend quadrangulair systeem wordt in dit geval genoteerd als $\Omega_D(K, K_0, L_0)$.
- De quadrangulaire systemen van **pseudokwadratisch type** worden geconstrueerd vanuit een willekeurige anisotrope pseudokwadratische ruimte (K, K_0, σ, V, p); het corresponderend quadrangulair systeem wordt in dit geval genoteerd als $\Omega_P(K, K_0, \sigma, V, p)$.
- De quadrangulaire systemen van **type E_6, E_7 en E_8** worden geconstrueerd vanuit een willekeurige anisotrope kwadratische ruimte (K, V, q) van type E_6, E_7 en E_8 respectievelijk; het corresponderend quadrangulair systeem wordt in dit geval genoteerd als $\Omega_E(K, V, q)$.
- De quadrangulaire systemen van **type F_4** worden geconstrueerd vanuit een willekeurige anisotrope kwadratische ruimte (K, V, q) van type F_4; het corresponderend quadrangulair systeem wordt in dit geval genoteerd als $\Omega_F(K, V, q)$.

In sectie 2.7 bewijzen we precies dat elk quadrangulair systeem tot (minstens) één van deze types behoort. We gebruiken daarbij de volgende opdeling.

Definitie B.2.8. Een quadrangulair systeem $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ wordt *indifferentieel* genoemd als $F \equiv 0$ en $H \equiv 0$, *gereduceerd* als $F \not\equiv 0$ en $H \equiv 0$, en *wijd* als $F \not\equiv 0$ en $H \not\equiv 0$.

Opmerking B.2.9. Als $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ een quadrangulair systeem is met $F \equiv 0$ en $H \not\equiv 0$, dan is $\Omega^* := (W, V, \delta, \epsilon)$ een gereduceerd quadrangulair systeem.

Definitie B.2.10. Zij $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ een wijd quadrangulair systeem. Stel $Y := \mathrm{Rad}(H)$. Dan zal $\Gamma := (V, Y, \epsilon, \delta)$ een gereduceerd quadrangulair systeem zijn. We zeggen dan dat Ω een *extensie* is van Γ.

Definitie B.2.11. Zij $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ een gereduceerd quadrangulair systeem. Dan wordt Ω *normaal* genoemd als en slechts als er voor alle elementen $w_1, w_2, \ldots, w_i \in W$ een element $w \in W$ bestaat zodanig dat $\epsilon w_1 w_2 \ldots w_i = \epsilon w$.

Veronderstel nu dat $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ een willekeurig quadrangulair systeem is. De klassificatie kan onderverdeeld worden in de volgende vijf stellingen.

Stelling B.2.12. *Als Ω gereduceerd is maar niet normaal is, dan is $\Omega \cong \Omega_I(K, K_0, \sigma)$ voor een zeker involutiesysteem (K, K_0, σ) zodanig dat $\sigma \neq 1$ en zodanig dat K voortgebracht is door K_0 als een ring.*

Stelling B.2.13. *Als Ω normaal is, dan is $\Omega \cong \Omega_Q(K, V_0, q)$ voor een zekere anisotrope kwadratische ruimte (K, V_0, q).*

Stelling B.2.14. *Als Ω indifferentieel is, dan is $\Omega \cong \Omega_D(K, K_0, L_0)$ voor een zeker indifferentiesysteem (K, K_0, L_0).*

Stelling B.2.15. *Als Ω een extensie is van een gereduceerd quadrangulair systeem $\Gamma = \Omega_I(K, K_0, \sigma)$ voor een zeker involutiesysteem (K, K_0, σ) zodanig dat $\sigma \neq 1$ en zodanig dat K voortgebracht is door K_0 als een ring, dan is $\Omega \cong \Omega_P(K, K_0, \sigma, V_0, p)$ voor een zekere anisotrope pseudokwadratische ruimte (K, K_0, σ, V_0, p).*

Stelling B.2.16. *Als Ω een extensie is van een gereduceerd quadrangulair systeem $\Gamma = \Omega_Q(K, V_0, q)$ voor een zekere anisotrope kwadratische ruimte (K, V_0, q), dan hebben we één van volgende gevallen:*

- *Er bestaat*

  (a) *een vermenigvuldiging op V_0 die van de vectorruimte V_0 een K-algebra maakt, zodanig dat ofwel V_0 een veld is, en V_0/K dan een separabele kwadratische uitbreiding met norm q, ofwel V_0 een quaternionen delingsalgebra is over K met norm q,*
  (b) *een involutie σ van V_0 (die het unieke niet-triviale element van $\mathrm{Gal}(V_0/K)$ is als $\dim_K V_0 = 2$ en die de standaard involutie van V_0 is als $\dim_K V_0 = 4$),*
  (c) *een niet-triviale rechtse vectorruimte X over V_0,*
  (d) *een pseudokwadratische vorm π over X,*

  zodanig dat (V_0, K, σ, X, π) een anisotrope pseudokwadratische ruimte is, $\Gamma \cong \Omega_I(V_0, K, \sigma)$ en $\Omega \cong \Omega_P(V_0, K, \sigma, X, \pi)$.
- *(K, V_0, q) is een kwadratische ruimte van type E_6, E_7 of E_8, en $\Omega \cong \Omega_E(K, V_0, q)$.*
- *(K, V_0, q) is een kwadratische ruimte van type F_4, en $\Omega \cong \Omega_F(K, V_0, q)$.*

Hoofdstuk 2 wordt afgesloten met een herformulering van het axiomasysteem voor het geval van *abelse* quadrangulaire systemen, dit zijn quadrangulaire systemen waarvoor niet alleen de groep V maar ook de groep W abels is. Zie pagina's 131 en volgende.

B.3 Automorfismen van F_4-vierhoeken

Een belangrijk probleem in de theorie van de Moufang veelhoeken is het bepalen van de structuur van de automorfismengroep G modulo de deelgroep $G^\dagger$ voortgebracht door alle wortelgroepen. In [52] is dit probleem opgelost voor vier van de zes klassen van Moufang vierhoeken. De twee gevallen die zijn opengelaten zijn deze van de exceptionele Moufang vierhoeken (die van type E_k en die van type F_4). De doelstelling van hoofdstuk 3 van deze thesis is dan ook om dit probleem op te lossen voor het geval van de F_4-vierhoeken. Meer bepaald tonen we aan dat de automorfismengroep op veldautomorfismen na volledig wordt voortgebracht door de wortelgroepen. Om dit resultaat te bereiken hebben we gebruik gemaakt van de quadrangulaire systemen die we in hoofdstuk 2 hebben ingevoerd.

Veronderstel dus dat $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ een quadrangulair systeem van type F_4 is, geconstrueerd zoals in sectie 2.6.6 vanuit de kwadratische ruimten (K, V, q) en $(L, W, \hat{q})$ met $L \leq K$. Zij $\Gamma := \mathcal{Q}(\Omega)$ de corresponderende Moufang vierhoek, en zij $G := \mathrm{Aut}(\Gamma)$ en $G^\dagger$ de deelgroep van G voortgebracht door alle wortelgroepen van Γ. Noteer de groep van veldautomorfismen van K die L op zichzelf afbeelden als $\mathrm{Aut}(K, L)$. Dan kunnen we het resultaat van hoofdstuk 3 als volgt neerschrijven.

Stelling B.3.1. *$G/G^\dagger$ is isomorf met een deelgroep van $\mathrm{Aut}(K, L)$.*

De eerste stap hierin is het vertalen van het meetkundig probleem naar een algebraïsch probleem.

Definitie B.3.2. Zij $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ een willekeurig quadrangulair systeem van type F_4. We noemen het koppel $(\varphi, \hat{\varphi})$ een *zelfgelijkvormigheid* van Ω als φ en $\hat{\varphi}$ groepsautomorfismen zijn van $(V, +)$ en van $(W, +)$, respectievelijk, zodanig dat er constanten $g \in K^*$ en $\hat{g} \in L^*$ bestaan (de *parameters* genoemd), zodat

$$\varphi(vw) = g\varphi(v)\hat{\varphi}(w)$$
$$\hat{\varphi}(wv) = \hat{g}\hat{\varphi}(w)\varphi(v)$$

voor alle $v \in V$ en alle $w \in W$. Een zelfgelijkvormigheid $(\varphi, \hat{\varphi})$ wordt *lineair* genoemd als en slechts als zowel φ als $\hat{\varphi}$ vectorruimte-isomorfismen zijn. Noteer de groep van alle zelfgelijkvormigheden als X, en zijn deelgroep van alle lineaire zelfgelijkvormigheden als X_ℓ.

Beschouw nu voor alle $c \in V^*$ de zelfgelijkvormigheid

$$\theta_c : (v, w) \mapsto (q(c)\pi_c(v), wc^{-1}) \ ,$$

en voor alle $z \in W^*$ de zelfgelijkvormigheid

$$\hat{\theta}_z : (v, w) \mapsto (vz^{-1}, \hat{q}(z)\hat{\pi}_z(w)) \ ,$$

en stel $X^\dagger := \langle \hat{\theta}_z, \theta_c \mid z \in W^*, c \in V^* \rangle$. Dan hebben we de volgende stelling.

Stelling B.3.3. $G/G^\dagger \cong X/X^\dagger$.

De rest van het bewijs splitsen we op in verschillende stappen.

- Het bepalen van $G(q)$, de groep van de multiplicatoren van de similitudes van q. Er blijkt dat $G(q) = K^2 \cdot \hat{q}(W) \cdot \hat{q}(W) \setminus \{0\}$.
- Reductie tot lineaire zelfgelijkvormigheden. Daartoe tonen we aan dat X/X_ℓ isomorf is met een deelgroep van $\mathrm{Aut}(K, L)$. Er rest ons nog aan te tonen dat $X_\ell = X^\dagger$. Voor het vervolg beschouwen we dus een willekeurige lineaire zelfgelijkvormigheid $(\varphi, \hat{\varphi}) \in X_\ell$ met parameters $(g, \hat{g})$.
- Reductie tot het geval $\hat{g} = 1$. Hiervoor hebben we de expliciete gedaante van $G(q)$ nodig.
- Reductie tot het geval $\varphi = 1$. Om dit te verkrijgen bewijzen we een stelling van het type Dieudonné-Cartan.
- Bepaling van $\hat{\varphi}$. We tonen tenslotte aan dat $g \in L^*$, en we besluiten dat daaruit volgt dat $(\varphi, \hat{\varphi}) \in X^\dagger$.

Hiermee is het bewijs van de hoofdstelling van dit hoofdstuk voltooid.

B.4 Kwadratische vormen van type E_6, E_7 en E_8

Uit B.2.10 zien we dat een wijd quadrangulair systeem op een kanonieke wijze een gereduceerd sub-quadrangulair systeem $\Gamma = \Gamma_\Omega$ bevat. Het geval waarin Γ een quadrangulair systeem van kwadratisch type is, corresponderende met een anisotrope kwadratische ruimte (K, V_0, q), is onderzocht in Stelling B.2.16; deze stelling leert ons ondermeer dat, als q regulier is en dimensie groter dan 4 heeft, dan is q een kwadratische vorm van type E_6, E_7 of E_8, en Ω is volledig bepaald door Γ en bijgevolg door q. Anderzijds volgt er uit de resultaten van [43] – zie ook de appendix van [52] voor meer details – dat, voor elke anisotrope kwadratische vorm q (over een veld K) waarvan de structuur van de even Clifford algebra wordt gegeven door Stelling B.4.1 hieronder, er een K-vorm van een enkelvoudige algebraïsche groep van type E_k bestaat waarvan het corresponderende sferische gebouw een wijde Moufang vierhoek Ω is zodanig dat Γ_Ω de vierhoek is bepaald door q. Deze resultaten impliceren samen dat de enige kwadratische vormen waarvan de even Clifford algebra is zoals in Stelling B.4.1 precies de kwadratische vormen van type E_6, E_7 en E_8 zijn. In hoofdstuk 4 geven we een rechtstreeks bewijs van dit resultaat. Tegelijk geeft ons dit meer inzicht in deze kwadratische vormen, en verkrijgen we ook nieuwe algemene resultaten over laag-dimensionale kwadratische vormen.

Zij (K, V, q) een willekeurige kwadratische vorm met $V \neq 0$. We noteren de discriminant van q als $d(q)$, en we noteren de Clifford invariant of de Witt invariant van q als $c(q) \in \mathrm{Br}(K)$. Voor even-dimensionale kwadratische vormen hebben we dus dat $c(q) = [C(q)] \in \mathrm{Br}(K)$.

Het doel van hoofdstuk 4 is om de kwadratische vormen van type E_6, E_7 en E_8 volledig te karakteriseren door middel van hun even Clifford algebra, of ook nog door middel van hun discriminant en hun Clifford invariant. Meer bepaald bewijzen we de volgende stelling.

Stelling B.4.1.

(i) *Zij q een 6-dimensionale anisotrope kwadratische vorm over K. Dan zijn de volgende uitspraken equivalent.*

  (a) *q is van type E_6.*
  (b) $C_0(q) \cong \mathrm{Mat}_4(E)$ *voor een zekere kwadratische uitbreiding E/K.*
  (c) *q is regulier, $d(q)$ is niet triviaal en $c(q) = [Q]$ voor een zekere quaternionen-algebra Q over K die, als het een delingsalgebra is, de discriminantuitbreiding van q bevat.*

(ii) *Zij q een 8-dimensionale anisotrope kwadratische vorm over K. Dan zijn de volgende uitspraken equivalent.*

  (a) *q is van type E_7.*
  (b) $C_0(q) \cong \mathrm{Mat}_4(D) \oplus \mathrm{Mat}_4(D)$ *voor een zekere quaternionen delingsalgebra D over K.*
  (c) *q is regulier, $d(q)$ is triviaal en $c(q) = [D]$ voor een zekere quaternionen delingsalgebra D over K.*

(iii) *Zij q een 12-dimensionale anisotrope kwadratische vorm over K. Dan zijn de volgende uitspraken equivalent.*

  (a) *q is van type E_8.*
  (b) $C_0(q) \cong \mathrm{Mat}_{32}(K) \oplus \mathrm{Mat}_{32}(K)$.
  (c) *q is regulier, $d(q)$ is triviaal en $c(q) = 0$.*

Het volgende eenvoudige lemma is essentieel in het bewijzen van deze stelling. Het laat namelijk toe om de dimensie te verlagen door "op te tensoren" met een kwadratische uitbreiding.

Lemma B.4.2. *Zij q een willekeurige even-dimensionale reguliere kwadratische vorm over K, met een decompositie $q \simeq q_1 \perp q'$, waarbij q_1 een 2-dimensionale reguliere kwadratische vorm is.*

(i) *Veronderstel dat de discriminant $d(q_1)$ triviaal is. Dan is $d(q') = d(q)$ en $c(q') = c(q_1) + c(q)$.*

(ii) *Veronderstel dat $d(q_1)$ niet triviaal is. Beschouw de discriminantuitbreiding E/K van q_1. Dan is $d(q'_E) = d(q_E)$ en $c(q'_E) = c(q_E)$.*

Met behulp van dit lemma kunnen we nu de dimensie stap voor stap verhogen, en we bewijzen achtereenvolgens de volgende uitspraken.

Stelling B.4.3.

(i) *Een 4-dimensionale reguliere kwadratische vorm q over K met triviale discriminant $d(q)$ heeft een norm splijting of is hyperbolisch.*

(ii) *Een 4-dimensionale reguliere kwadratische vorm q over K met triviale discriminant $d(q)$ en met $c(q) = 0$ is hyperbolisch.*

(iii) *Een 6-dimensionale reguliere kwadratische vorm q over K met triviale discriminant $d(q)$ en met $c(q) = 0$ is hyperbolisch.*

(iv) *Zij q een 6-dimensionale anisotrope reguliere kwadratische vorm over K met niet-triviale discriminant $d(q)$. Beschouw de discriminantuitbreiding E/K van q. Als $c(q_E) = 0$ in $\mathrm{Br}(E)$, dan heeft q een norm splijting.*

(v) *Een 6-dimensionale reguliere kwadratische vorm q over K met triviale discriminant $d(q)$ en met $c(q) = [Q]$ voor een zekere quaternionenalgebra Q over K, is isotroop.*

(vi) *Een 8-dimensionale anisotrope reguliere kwadratische vorm q over K met triviale discriminant $d(q)$ en met $c(q) = [Q]$ voor een zekere quaternionenalgebra Q over K, heeft een norm splijting.*

(vii) *Een 10-dimensionale reguliere kwadratische vorm q over K met triviale discriminant $d(q)$ en met $c(q) = 0$ is isotroop.*

(viii) *Een 12-dimensionale anisotrope reguliere kwadratische vorm q over K met triviale discriminant $d(q)$ en met $c(q) = 0$ heeft een norm splijting.*

Merk op dat we al deze stellingen bewezen hebben voor velden van willekeurige karakteristiek. Ook hebben we bewezen dat we in de gedeelten (b) van Stelling B.4.1 de regulariteit niet hoeven te eisen, maar dat deze volgt uit de structuur van de even Clifford algebra. Meer algemeen hebben we de volgende stelling (die eigenlijk enkel betekenis heeft voor velden van karakteristiek 2).

Stelling B.4.4.

(i) *Als q_1 en q_2 twee anisotrope kwadratische vormen zijn van dezelfde dimensie, en als $C_0(q_1) \cong C_0(q_2)$, dan zijn q_1 en q_2 ofwel beide regulier, ofwel beide singulier.*

(ii) *Als q_1 en q_2 twee anisotrope kwadratische vormen zijn van dezelfde dimensie, en als $C(q_1) \cong C(q_2)$, dan zijn q_1 en q_2 ofwel beide regulier, ofwel beide singulier.*

Voor meer details verwijzen we – hoe kan het ook anders – naar het Engelstalige gedeelte van deze thesis.

Bibliography

[1] **A. A. Albert**: On the Wedderburn norm condition for cyclic algebras, *Bull. Amer. Math. Soc.* **37** (1931), 301–312.

[2] ______: On exceptional Jordan division algebras, *Pacific J. Math.* **15** (1965), 377–404.

[3] ______: Tensor products of quaternion algebras, *Proc. Amer. Math. Soc.* **35** (1972), 65–66.

[4] **R. Baer**: Homogeneity of projective planes, *Amer. J. Math.* **64** (1942), 137–152.

[5] **J.C. Baez**: The octonions, *Bull. Amer. Math. Soc. (N.S.)* **39** (2002), 145–205.

[6] **R. Brauer**: Über die algebraische Struktur von Schiefkörpern, *J. Reine Angew. Math.* **166** (1932), 241–252.

[7] **R. Brauer**: "Collected papers," ed. by P. Fong and W.J. Wong, MIT press, Cambridge (Mass.), 1980.

[8] **R. Brauer**, **H. Hasse** and **E. Noether**: Beweis eines Hauptsatzes in der Theorie der Algebren, *J. Reine Angew. Math.* **167** (1931), 399–404.

[9] **K.S. Brown**: "Buildings," Springer-Verlag, New York, 1989.

[10] **R. H. Bruck** and **E. Kleinfeld**: The structure of alternative division rings, *Proc. Amer. Math. Soc.* **2** (1951), 878–890.

[11] **W.K. Clifford**: Applications of Grassmann's extensive algebra, *Amer. J. Math.* **1** (1878), 350–358.

[12] **H.S.M. Coxeter**: The complete enumeration of finite groups of the form $R_i^2 = (R_iR_j)^{k_{ij}} = 1$, *J. London Math. Soc.* **10** (1935), 21–25.

[13] **T. De Medts**: A characterization of quadratic forms of type E_6, E_7 and E_8, *J. Algebra* **252** (2002), 394–410.

[14] ______: An Algebraic Structure for Moufang Quadrangles, *submitted*.

[15] ______: Automorphisms of F_4 quadrangles, *in preparation*.

[16] **P. Draxl**: Über gemeinsame separabel-quadratische Zerfällungskörper von Quaternionenalgebren, *Nachr. Akad. Wiss. Göttingen, Math. Phys. Kl. II* **16** (1975), 251–259.

[17] **W. Feit** and **G. Higman**: The nonexistence of certain generalized polygons, *J. Algebra* **1** (1964), 114–131.

[18] **P. Fong** and **G. Seitz**: Groups with a BN-pair of rank 2, I, *Invent. Math.* **21** (1973), 1–57.

[19] ______: Groups with a BN-pair of rank 2, II, *Invent. Math.* **24** (1974), 191–239.

[20] **A. J. Hahn**: "Quadratic Algebras, Clifford Algebras and Arithmetic Witt Groups," Springer-Verlag Berlin, Heidelberg, New York, 1994.

[21] **M. Hall, Jr.**: Projective Planes, *Trans. Amer. Math. Soc.* **54** (1943), 229–277.

[22] **D.R. Hughes** and **F.C. Piper**: "Projective Planes," Springer-Verlag Berlin, Heidelberg, New York, 1973.

[23] **F. D. Jacobson** and **N. Jacobson**: Classification and representation of semisimple Jordan algebras, *Trans. Amer. Math. Soc.* **65** (1949), 141–169.

[24] **N. Jacobson**: Structure and Representation of Jordan Algebras, *Amer. Math. Soc. Colloquium Ser.* **39**, Providence, 1968.

[25] ______: "Lectures on Quadratic Jordan Algebras," Tata Institute Lecture Notes, 1969.

[26] **N. Jacobson** and **K. McCrimmon** : Quadratic Jordan algebras of quadratic forms with basepoint, *J. Indian Math. Soc.* **35** (1971), 1–45.

[27] **M.-A. Knus**: Sur la forme d'Albert et le produit tensoriel de deux algèbres de quaternions, *Bull. Soc. Math. Belg. Sér. B* **45** (1993), no. 3, 333–337.

[28] **M.-A. Knus**, **A. S. Merkurjev**, **H. M. Rost** and **J.-P. Tignol**: "The Book of Involutons," Amer. Math. Soc. Colloquium Series, Providence, 1998.

[29] **T. Y. Lam**: "The Algebraic Theory of Quadratic Forms," Benjamin, Reading, 1973.

[30] ______: On the linkage of quaternion algebras, *Bull. Belg. Math. Soc. Simon Stevin* **9** (2002), no. 3, 415–418.

[31] **K. McCrimmon**: The Freudenthal-Springer-Tits constructions of exceptional Jordan algebras, *Trans. Amer. Math. Soc.* **139** (1969), 495–510.

[32] ______: The Freudenthal-Springer-Tits constructions revisited, *Trans. Amer. Math. Soc.* **148** (1970), 293–314.

[33] **R. Moufang**: Alternativkörper und der Satz vom vollständigen Vierseit, *Abh. Math. Sem. Univ. Hamburg* **9** (1933), 207–222.

[34] **B. Mühlherr** and **H. Van Maldeghem**: Exceptional Moufang quadrangles of type F_4, *Canad. J. Math.* **51** (1999), 347–371.

[35] **S.E. Payne** and **J.A. Thas**: "Finite Generalized Quadrangles," Pitman Res. Notes Math. Ser. **110**, Longman, London, Boston, Melbourne, 1984.

[36] **H. P. Petersson** and **M. L. Racine**: Jordan algebras of degree 3 and the Tits process, *J. Algebra* **98** (1986), 211–243.

[37] ______: Classification of algebras arising from the Tits process, *J. Algebra* **98** (1986), 244–279.

[38] **G. Pickert**: "Projektive Ebenen," Springer-Verlag, Berlin, 1955.

[39] **M.A. Ronan**: "Lectures on Buildings," Academic Press, San Diego, *Persp. Math.* **7**, 1989.

[40] **C. H. Sah**: Symmetric bilinear forms and quadratic forms, *J. Algebra* **20** (1972), 144–160.

[41] **W. Scharlau**: "Quadratic and Hermitian Forms," *Grundlehren der mathematischen Wissenschaften* **270**, Springer-Verlag Berlin, Heidelberg, New York, Tokyo, 1985.

[42] **J. Tits**: Sur la trialité et certains groupes qui s'en déduisent, *Inst. Hautes Études Sci. Publ. Math.* **2** (1959), 13–60.

[43] ______: Classification of algebraic semisimple groups, *in* "Algebraic Groups and Discontinuous Groups," Boulder 1965, *Proc. Symp. Pure Math.* **9** (1966), 33–62.

[44] ______: *Buildings of Spherical Type and Finite BN-Pairs*, Lecture Notes in Mathematics **386**, Springer-Verlag, New York/Heidelberg/Berlin, 1974.

[45] ______: Classification of buildings of spherical type and Moufang polygons: a survey, **in** *Coll. Intern. Teorie Combinatorie Acc. Naz. Lincei*, Proceedings Roma 1973, *Atti dei Convegni Lincei* **17** (1976), 229–246.

[46] ______: Non-existence de certains polygons généralisés, I, *Invent. Math.* **36** (1976), 275–284.

[47] ______: Endliche Spiegelungsgruppen, die als Weylgruppen auftreten, *Invent. Math.* **43** (1977), 283–295.

[48] ______: Non-existence de certains polygons généralisés, II, *Invent. Math.* **51** (1979), 267–269.

[49] ______: Moufang octagons and Ree groups of type ${}^2F_4$, *Amer. J. Math.* **105** (1983), 539–594.

[50] ______: Sur les produits tensoriels de deux algèbres de quaternions, *Bull. Soc. Math. Belg. Sér. B* **45** (1993), no. 3, 329–331.

[51] ______: Moufang polygons, I. Root Data, *Bull. Belg. Math. Soc. Simon Stevin* **1** (1994), 455–468.

[52] **J. Tits** and **R. Weiss**: "Moufang Polygons," Springer Monographs in Mathematics, Springer-Verlag, Berlin/Heidelberg/New York, 2002.

[53] **H. Van Maldeghem**: "Generalized Polygons," Monographs in Mathematics **93**, Birkhaüser Verlag, Basel/Boston/Berlin, 1998.

[54] **R. Weiss**: The nonexistence of certain Moufang polygons, *Invent. Math.* **51** (1979), 261–266.

[55] **M. Zorn**: Theorie der alternativen Ringe, *Abh. Math. Sem. Hamburg. Univ.* **8** (1930), 123–147.

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